Solution to assignment 10

(1) (16.5, Q19):
Use the cylindrical coordinates. Then a parametrization of the surface is

$$\mathbf{r}(r,\theta) = (r\cos\theta, r\sin\theta, 2r), \quad r \in (1,3), \theta \in (0,2\pi).$$

$$\mathbf{r}_r = (\cos\theta, \sin\theta, 2)$$
$$\mathbf{r}_\theta = (-r\sin\theta, r\cos\theta, 0)$$
$$\mathbf{r}_r \times \mathbf{r}_\theta = (-2r\cos\theta, -2r\sin\theta, r)$$
$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{5}r$$

So the area is $\int_0^{2\pi}\int_1^3 \sqrt{5}r\,dr\,d\theta = 8\sqrt{5}\pi$.

(2) (16.5, Q33):
(a) If we use the spherical coordinates, then the parametrization should be

$$\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta, b\sin\phi\sin\theta, c\cos\phi)$$

where $\phi \in [0,\pi]$, $\theta \in [0,2\pi]$.
(b)

$$\mathbf{r}_\phi = (a\cos\phi\cos\theta, b\cos\phi\sin\theta, -c\sin\phi)$$
$$\mathbf{r}_\theta = (-a\sin\phi\sin\theta, b\sin\phi\cos\theta, 0)$$
$$\mathbf{r}_\phi \times \mathbf{r}_\theta = (bc\sin^2\phi\cos\theta, ac\sin^2\phi\sin\theta, ab\sin\phi\cos\phi)$$
$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{b^2c^2\sin^4\phi\cos^2\theta + a^2c^2\sin^4\phi\sin^2\theta + a^2b^2\sin^2\phi\cos^2\phi}$$

So the area is

$$\int_0^{2\pi}\int_0^{\pi}\sqrt{b^2c^2\sin^4\phi\cos^2\theta + a^2c^2\sin^4\phi\sin^2\theta + a^2b^2\sin^2\phi\cos^2\phi}\,d\phi\,d\theta.$$

(3) (16.5, Q48):

$$\nabla f = 3\sqrt{x}\mathbf{i} + 3\sqrt{y}\mathbf{j} - 3\mathbf{k}$$
$$|\nabla f| = \sqrt{9x+9y+9} = 3\sqrt{x+y+1}$$
$$\mathbf{p} = \mathbf{k}, |\nabla f \cdot \mathbf{p}| = 3$$

$S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA$
$= \iint_R \sqrt{x+y+1}\,dx\,dy$
$= \int_0^1\int_0^1 \sqrt{x+y+1}\,dx\,dy$
$= \int_0^1 \left[\frac{2}{3}(x+y+1)^{3/2}\right]_0^1 dy$
$= \int_0^1 \left[\frac{2}{3}(y+2)^{3/2} - \frac{2}{3}(y+1)^{3/2}\right]_0^1 dy$

$= \left[\frac{4}{15}(y+2)^{5/2} - \frac{4}{15}(y+1)^{5/2}\right]_0^1$
$= \frac{4}{15}\left[(3)^{5/2} - (2)^{5/2} - (2)^{5/2} + 1\right]$
$= \frac{4}{15}(9\sqrt{3} - 8\sqrt{2} + 1).$

(4) (16.5, Q56):
(a) Let (x, y, z) be a point on S. Consider the cross section when $x = x^*$, the cross section is a circle with radius $r = f(x^*)$. The set of parametric equations for this circle are given by $y(\theta) = r\cos\theta = f(x^*)\cos\theta$ and $z(\theta) = r\sin\theta = f(x^*)\sin\theta$ where $\theta \in [0, 2\pi]$.
Since x can take on any value between a and b we have $x(x,\theta) = x, y(x,\theta) = f(x)\cos\theta, z(x,\theta) = f(x)\sin\theta$ where $x \in [a, b]$ and $\theta \in [0, 2\pi]$. Thus

$$\mathbf{r}(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta).$$

(b)

$$\mathbf{r}_x = (1, f^{'}(x)\cos\theta, f^{'}(x)\sin\theta)$$
$$\mathbf{r}_\theta = (0, -f(x)\sin\theta, f(x)\cos\theta)$$
$$\mathbf{r}_x \times \mathbf{r}_\theta = (f(x)f^{'}(x), -f(x)\cos\theta, -f(x)\sin\theta)$$
$$|\mathbf{r}_x \times \mathbf{r}_\theta| = f(x)\sqrt{1 + (f'(x))^2}$$

So the area is

$$\int_0^{2\pi}\int_a^b f(x)\sqrt{1+(f'(x))^2}dxd\theta = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}dx.$$

(5) (16.6, Q10):
On the face S in the xz-plane, we have

$$y = 0 \Rightarrow f(x,y,z) = y = 0$$
$$G(x,y,z) = G(x,0,z) = z$$
$$\mathbf{p} = \mathbf{j}, \nabla f = \mathbf{j}, |\nabla f| = 1, |\nabla f \cdot \mathbf{p}| = 1$$
$$d\sigma = dxdz$$
$$\iint_S G d\sigma = \iint_S (y+z)d\sigma = \int_0^1\int_0^2 z dxdz = \int_0^1 2z dz = 1$$

On the face in the xy-plane, we have

$$z = 0 \Rightarrow f(x,y,z) = z = 0$$
$$G(x,y,z) = G(x,y,0) = y$$
$$\mathbf{p} = \mathbf{k}, \nabla f = \mathbf{k}, |\nabla f| = 1, |\nabla f \cdot \mathbf{p}| = 1$$
$$d\sigma = dxdy$$

$$\iint_S G d\sigma = \iint_S y d\sigma = \int_0^1 \int_0^2 y dx dy = 1.$$

On the triangular face in the plane $x = 2$ we have

$$f(x, y, z) = x = 2$$
$$G(x, y, z) = G(2, y, z) = y + z$$
$$\mathbf{p} = \mathbf{i}, \nabla f = \mathbf{i}, |\nabla f| = 1, |\nabla f \cdot \mathbf{p}| = 1$$
$$d\sigma = dz dy$$
$$\iint_S G d\sigma = \iint_S (y+z) d\sigma = \int_0^1 \int_0^{1-y} (y+z) dz dy = \int_0^1 \frac{1}{2}\left(1 - y^2\right) dy = \frac{1}{3}.$$

On the triangular face in the yz-plane we have

$$x = 0 \Rightarrow f(x, y, z) = x = 0$$
$$G(x, y, z) = G(0, y, z) = y + z$$
$$\mathbf{p} = \mathbf{i}, \nabla f = \mathbf{i}, |\nabla f| = 1, |\nabla f \cdot \mathbf{p}| = 1$$
$$d\sigma = dz dy$$
$$\iint_S G d\sigma = \iint_S (y+z) d\sigma = \int_0^1 \int_0^{1-y} (y+z) dz dy = \frac{1}{3}.$$

Finally, on the sloped face, we have

$$y + z = 1 \Rightarrow f(x, y, z) = y + z = 1$$
$$G(x, y, z) = y + z = 1$$
$$\mathbf{p} = \mathbf{k}, \nabla f = \mathbf{j} + \mathbf{k}, |\nabla f| = \sqrt{2}, |\nabla f \cdot \mathbf{p}| = 1$$
$$d\sigma = \sqrt{2} dx dy$$
$$\iint_S G d\sigma = \iint_S (y+z) d\sigma = \int_0^1 \int_0^2 \sqrt{2} dx dy = 2\sqrt{2}.$$

Therefore, $\iint_{\text{wedge}} G(x, y, z) d\sigma = 1 + 1 + \frac{1}{3} + \frac{1}{3} + 2\sqrt{2} = \frac{8}{3} + 2\sqrt{2}.$

Selection solution to Assignment 10

Supplementary Problems

1. Let $(x(t), y(t)),\ t \in [a, b]$, be a curve C parametrized by t in the first quadrant. Rotate it around the x-axis to get a surface of revolution S.

   (a) Show that a parametrization of S is given by $(\alpha, t) \mapsto (x(t), y(t)\cos\alpha, y(t)\sin\alpha)$ $\alpha \in [0, 2\pi]$, and it is regular when C is regular.

   Solution. By a direct computation,

$$\frac{\partial \mathbf{r}}{\partial \alpha} = (0, -y\sin\alpha, y\cos\alpha),$$

$$\frac{\partial \mathbf{r}}{\partial t} = (x', y'\cos\alpha, y'\sin\alpha),$$

so

$$\frac{\partial \mathbf{r}}{\partial \alpha} \times \frac{\partial \mathbf{r}}{\partial t} = (-yy', x'y\cos\alpha, -yx'\sin\alpha) \ ,$$

and

$$\left|\frac{\partial \mathbf{r}}{\partial \alpha} \times \frac{\partial \mathbf{r}}{\partial t}\right| = |y|\sqrt{x'^2 + y'^2} \ .$$

   When C is regular, that is, $x'^2 + y'^2 > 0$, it is clear that S is also regular.

   (b) Show that the surface area of S is given by

$$2\pi \int_C y(t)\, ds \ .$$

   Solution. The parameter (α, t) lies in $D = [0, 2\pi] \times [a, b]$. By the surface area formula, the surface area of S is equal to

$$\begin{aligned}
\iint_D y\sqrt{x'^2(t) + y'^2(t)}\, dA &= \int_0^{2\pi}\int_a^b y(t)\sqrt{x'^2(t) + y'^2(t)}\, dt d\alpha \\
&= 2\pi \int_a^b y(t)\sqrt{x'^2(t) + y'^2(t)}\, dt \\
&= 2\pi \int_C y\, ds \ .
\end{aligned}$$

   (c) When $y = f(x), x \in [a, b]$, where f is C^1, the surface area becomes

$$2\pi \int_a^b f(x)\sqrt{1 + f'^2(x)}\, dx \ .$$

Solution. Here the parametrization of the curve C is given by $x \mapsto (x, f(x))$. Therefore, $y(x) = f(x), x'(x) = 1$ and $y'(x) = f'(x)$ and (c) follows from (b).