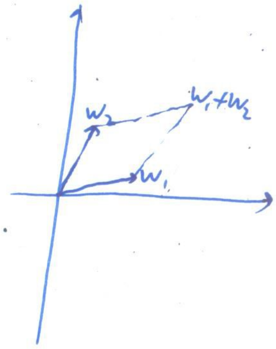
MATH 2230 Tutorial 9

1. Let f be a function on $\mathbb{C}$ and f is not a constand.
Suppose $\exists\, w_1, w_2 \in \mathbb{C}\setminus\{0\}$ such that

$$\begin{cases} \frac{w_1}{w_2} \notin R \\ f(z+w_1)=f(z) & \text{for any } z\in\mathbb{C} \\ f(z+w_2)=f(z) & \text{for any } z\in\mathbb{C} \end{cases}$$

Show that f can not be an entire function.

Pf: [prove by contradiction].



Suppose f is an entire function

$\because \frac{w_1}{w_2} \notin R$

$\therefore$ there is some closed area R which is bounded by $0, w_1, w_2, w_1+w_2$

$\because \begin{cases} f(z+w_1)=f(z) \\ f(z+w_2)=f(z) \end{cases}$

$\therefore \forall z\in\mathbb{C},\ \exists\, z_0\in R,\ k_1,k_2\in\mathbb{Z}$ such that $z=z_0+k_1w_1+k_2w_2$

$\therefore f(z)=f(z_0+k_1w_1+k_2w_2)=f(z_0)$

$\because f$ is analytic on R

$\therefore |f|$ is bounded on R

$\because \forall z\in\mathbb{C},\ \exists\, z_0\in R$ such that $f(z)=f(z_0)$

$\therefore |f|$ is bounded on $\mathbb{C}$

$\because f$ is entire

$\therefore$ By Liouville's Thm,

f is a constant

But this is a contradiction to our assumption that f is not a constant.

$\therefore f$ can not be an entire function.

Remark: This tells that
there is no non-constant, doubly periodic entire function.

2. Let f be an analytic function and not a constant

Show that the zeros of f are isolated.

i.e: If z_0 is a zero of f, then $\exists\, r>0$ such that

$f(z) \neq 0$ on $B_r(z_0) \setminus \{z_0\} := \{z_0 + ae^{i\theta} : a \leq r\} \setminus \{z_0\}$.

Pf: Let z_0 be a zero of f

$\because$ f is analytic at z_0

$\therefore \exists\, r>0$ such that

$f(z)$ is analytic on $B_r(z_0)$

$\therefore$ By Taylor's Thm

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n \quad \text{for } \forall z \in B_r(z_0)$$

$\because$ $f(z)$ is not a constant

$\therefore \exists\, n_0 \in N$ such that

$$\begin{cases} f^k(z_0) = 0 & \text{for } k = 0, 1, \cdots, n_0 - 1 \\ f^{n_0}(z_0) \neq 0 \end{cases}$$

$$\begin{aligned}\therefore f(z) &= \sum_{n=n_0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n \\ &= (z-z_0)^{n_0}\left[\frac{f^{n_0}(z_0)}{n_0!} + \frac{f^{n_0+1}(z_0)}{(n_0+1)!}(z-z_0) + \cdots\right] \\ &= (z-z_0)^{n_0}\left[\frac{f^{n_0}(z_0)}{n_0!} + g(z)\right]\end{aligned}$$

$\because$ $g(z)$ is analytic and $g(z_0) = 0$ [need to use the convergence of power series to show g is analytic].

$\therefore$ when r is sufficient small

$|g(z)|$ is small on $B_r(z_0)$

$\because$ $f^{n_0}(z_0) \neq 0$

$\therefore$ $\dfrac{f^{n_0}(z_0)}{n_0!} + g(z)$ is not zero on $B_r(z_0)$.

$\because$ $f(z) = (z-z_0)^n\left[\dfrac{f^{n_0}(z_0)}{n_0!} + g(z)\right]$ on $B_r(z_0)$

$\therefore$ $f(z) \neq 0$ for $\forall z \in B_r(z_0) \setminus \{z_0\}$

$\therefore$ the zeros of f are isolated.

3. Let f, g be two entire function

Suppose $\exists$ a curve γ such that $f(z) = g(z)$ on γ

Show $f \equiv g$ on $\mathbb{C}$.

Pf: Let $h(z) = f(z) - g(z)$

then $\begin{cases} h(z) \text{ is entire} \\ h(z) \equiv 0 \text{ on } \gamma \end{cases}$

If h is not a constant on $\mathbb{C}$

then by Q2, the zeros of h are isolated

But $h(z) \equiv 0$ on γ, which is a contradiction

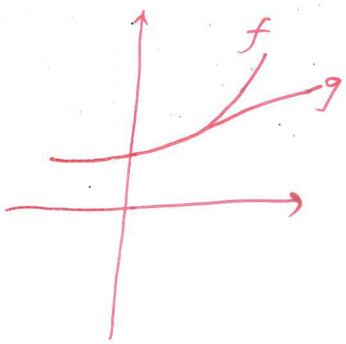
$\therefore$ h is a constant

$\therefore$ $h \equiv 0$ on $\mathbb{C}$

$\therefore$ $f \equiv g$ on $\mathbb{C}$

· Remark: In real case, we can have two different functions which are equal on an interval.

But in complex case, it is much more resenitive.



4. In the previous computation,

if f is analytic, non-constant; $f(z_0)=0$

then $f(z)=(z-z_0)^{n_0}\left[\frac{f^{n_0}(z_0)}{n_0!}+g(z)\right]$ on $B_r(z_0)$

$=(z-z_0)^{n_0}h(z)$

where $\begin{cases} h(z) \text{ is analytic on } B_r(z_0) \\ h(z) \text{ is not zero on } B_r(z_0) \end{cases}$

- we call n_0 the order of z_0 (or z_0 is a zero of f of order n_0)

5. Let f, g be two entire functions which satisfy

$$|f(z)| \leq 2|g(z)| \text{ for } \forall z \in \mathbb{C}$$

show $\exists\, C_0 \in \mathbb{C}$ such that $f = C_0 g$

Pf: Let $h(z) = \frac{f(z)}{g(z)}$.

1°: we show $h(z)$ is entire.

obviously, $h(z)$ is analytic at z where $g(z) \neq 0$

Let z_0 be a zero of $g(z)$

$\because |f| \leq 2|g| \quad \therefore f(z_0) = 0$

$\therefore \exists\, n_1, n_2, F, G$ such that

$$\begin{cases} f(z) = (z-z_0)^{n_1} F(z) \\ g(z) = (z-z_0)^{n_2} G(z) \end{cases} \text{ around } z_0$$

and F, G are analytic, non-zero around z_0

$\because |f| \leq 2|g|$

$\therefore |z-z_0|^{n_1} |F| \leq 2|z-z_0|^{n_2} |G|$

claim: $n_1 \geq n_2$

if $n_1 < n_2$, then $|F| \leq 2|z-z_0|^{n_2-n_1} |G|$ around z_0

but $F(z_0) \neq 0$, $\lim\limits_{z \to z_0} 2|z-z_0|^{n_2-n_1} |G| = 0$ since $n_2 > n_1$

it is a contradiction

$\therefore n_1 \geq n_2$

$\therefore h = \frac{f}{g} = \frac{(z-z_0)^{n_1} F}{(z-z_0)^{n_2} G} = (z-z_0)^{n_1-n_2} \cdot \frac{F}{G}$ is analytic at z_0

since $n_1 > n_2$, $G(z_0) \neq 0$

$\therefore h$ is entire

2°, $\because |f| \leq 2|g| \quad \therefore |h| = \frac{|f|}{|g|} \leq \frac{2|g|}{|g|} = 2$

$\therefore$ By Liouville's Thm

h is a constant

$\therefore h \equiv C_0$ for some $C_0 \in \mathbb{C}$

$\therefore f = C_0 g$ for some $C_0 \in \mathbb{C}$.