MATH 2230 Tutorial

(C-R eqn) $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

1. Let $f = u + iv$, where $u = 2xy$

Find v such that f is differentiable.

Ans: $\because u = 2xy$

$\therefore u_x = 2y, \quad u_y = 2x$

$\therefore$ By C-R eqn, $\begin{cases} v_x = -u_y = -2x \\ v_y = u_x = 2y \end{cases} \Rightarrow v = y^2 - x^2 + C$

$\therefore f = u + iv = 2xy + i(y^2 - x^2 + C)$

• Write f in terms of z

let $z = x + iy \Rightarrow x = z - iy$

$\therefore f = 2xy + i(y^2 - x^2) + Ci$

$= 2y(z - iy) + i[y^2 - (z - iy)^2] + Ci$

$= 2yz - 2iy^2 + i[y^2 - (z^2 - 2iyz - y^2)] + Ci$

$= 2yz - 2iy^2 + 2iy^2 + iz^2 - 2yz + Ci$

$= iz^2 + Ci$

2. Given $u(x,y) = xe^x\cos(y) - x - ye^x\sin(y)$

Find $v(x,y)$ such that $f = u + iv$ is differentiable.

Ans: Different method from Q1.

$\because u = xe^x\cos(y) - x - ye^x\sin(y)$

$\therefore u_x = e^x\cos y + xe^x\cos y - 1 - ye^x\sin y$

$u_y = -xe^x\sin y - e^x\sin y - ye^x\cos y$

If such f exists, then

$f' = u_x + iv_x = u_x - iu_y$

$\cancel{= xe^x\cos y - x - ye^x\sin y - i(-xe^x\sin y - e^x\sin y - ye^x\cos y)}$

$\cancel{= xe^x(\cos y + i\sin y)}$

$= e^x\cos y + xe^x\cos y - 1 - ye^x\sin y - i(-xe^x\sin y - e^x\sin y - ye^x\cos y)$

$= e^x(\cos y + i\sin y) + xe^x(\cos y + i\sin y)$

$\quad - ye^x(\sin y - i\cos y) - 1$

$= e^x e^{iy} + xe^x e^{iy} + iye^x e^{iy} - 1$

$= e^z + xe^z + iye^z - 1 = e^z + ze^z - 1$

$\therefore f(z) = ze^z - z + C$

3. Let $u = x^2 y$

Does there exist a v such that $f = u + iv$ is differentiable on whole $\mathbb{C}$?

Ans: $u_x = 2xy$, $u_{xx} = 2y$

$u_y = x^2$, $u_{yy} = 0$.

$\because u_{xx} + u_{yy} = 0 + 2y \neq 0$ for $y \neq 0$

$\therefore$ such v doesn't exist.

4. Find all z such that

i) $e^z = -2$

ii) $e^z = 1 + i$

iii) $e^{2z-1} = 1$

Ans: i) $-2 = 2 \cdot (-1) = e^{\lg 2} \cdot e^{i(\pi + 2n\pi)} = e^{\lg 2 + i(\pi + 2n\pi)}$

$\therefore z = \lg 2 + i(\pi + 2n\pi)$

ii) $1 + i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = e^{\lg\sqrt{2}} e^{i\left(\frac{\pi}{4} + 2n\pi\right)} = e^{\frac{1}{2}\lg 2 + \left(\frac{1}{4} + 2n\right)\pi i}$

$\therefore z = \frac{1}{2}\lg 2 + \left(\frac{1}{4} + 2n\right)\pi i$

iii) $1 = e^{i 2n\pi}$

$\therefore 2z - 1 = i2n\pi$

$z = \frac{1}{2} + n\pi i$

5、 sinz as a mapping:

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i} = \frac{e^{-y}e^{ix}-e^{y}e^{-ix}}{2i}$$

$$= \frac{e^{-y}(\cos x+i\sin x)-e^{y}(\cos x-i\sin x)}{2i}$$

$$= \frac{1}{2}(e^{y}+e^{-y})\sin x + \frac{1}{2i}(e^{-y}-e^{y})\cos x$$

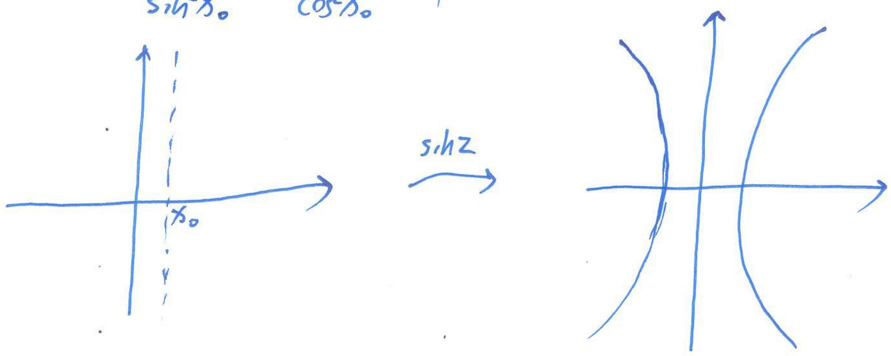
$$= \frac{1}{2}(e^{y}+e^{-y})\sin x + \frac{i}{2}(e^{y}-e^{-y})\cos x$$

$$\therefore u = \frac{1}{2}(e^{y}+e^{-y})\sin x$$

$$v = \frac{1}{2}(e^{y}-e^{-y})\cos x$$

• Fix $x=x_0$, if $\sin x_0 \neq 0$, $\cos x_0 \neq 0$

then $\frac{u^2}{\sin^2 x_0} - \frac{v^2}{\cos^2 x_0} = 1$



• Fix $y=y_0$, if $y_0 \neq 0$, then

$$\frac{u^2}{\frac{1}{4}(e^{y_0}+e^{-y_0})^2} + \frac{v^2}{\frac{1}{4}(e^{y_0}-e^{-y_0})^2} = 1$$