$\underline{\text{Thm 2}}$ Let $p(z)$, $q(z)$ be analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$ and $\underline{q'(z_0) \neq 0}$.

Then z_0 is a $\underline{\text{simple pole}}$ of $f(z) = \dfrac{p(z)}{q(z)}$

and
$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

$\underline{\text{Pf:}}$ Assumption $\Rightarrow$
$$q(z) = \cancelto{0}{q(z_0)} + q'(z_0)(z-z_0) + \frac{q''(z_0)}{2!}(z-z_0)^2 + \cdots$$

ie.
$$q(z) = (z-z_0)\left(q'(z_0) + \frac{q''(z_0)}{2!}(z-z_0) + \cdots \right)$$

$$\therefore \quad f(z) = \frac{p(z)}{q(z)} = \frac{1}{(z-z_0)}\left(\frac{p(z)}{g(z)}\right)$$

where $g(z) = q'(z_0) + \dfrac{q''(z_0)}{2!}(z-z_0) + \cdots$

is analytic & $g(z_0) = q'(z_0) \neq 0$

Together $p(z_0) \neq 0$, we have $\dfrac{p(z_0)}{g(z_0)} = \dfrac{p(z_0)}{q'(z_0)} \neq 0$

$\therefore$ z_0 is simple of $f(z)$ and
$$\operatorname{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} \quad ✗$$

eg: $f(z) = \cot z = \dfrac{\cos z}{\sin z}$

Let $p(z) = \cos z$, $q(z) = \sin z$

Then $\forall\, z = n\pi$, $n \in \mathbb{Z}$, $q(n\pi) = 0$

and $q'(n\pi) = \cos(n\pi) = (-1)^n \neq 0$

$\therefore$ $z = n\pi$, $\forall n \in \mathbb{Z}$, are simple poles of $\cot z$,

and $\operatorname*{Res}_{z=n\pi} \cot z = \dfrac{p(n\pi)}{q'(n\pi)} = \dfrac{\cos n\pi}{\cos n\pi} = 1$.

$\#$

eg: $f(z) = \dfrac{z - \sinh z}{z^2 \sinh z}$

Consider the pole $z = \pi i$

Let $p(z) = z - \sinh z$ & $q(z) = z^2 \sinh z$

Then $p(\pi i) = \pi i - 0 = \pi i \neq 0$

$q(\pi i) = (\pi i)^2 \sinh(\pi i) = 0$

$q'(\pi i) = \left[2z \sinh z + z^2 \cosh z\right]_{\pi i}$

$= (\pi i)^2 \cosh(\pi i) = \pi^2 \neq 0$

(check)

$\therefore$ $z = \pi i$ is a simple pole of $f(z)$ and

$$\operatorname*{Res}_{z=\pi i} f(z) = \frac{p(\pi i)}{q'(\pi i)} = \frac{\pi i}{\pi^2} = \frac{i}{\pi}$$

\#

§84 Behavior of the functions near isolated singular points

(a) Removable Singular Points

Thm 1 If z_0 is a removable singular point of f, then f is bounded and analytic in $0 < |z - z_0| < \varepsilon$ for some $\varepsilon > 0$.

Thm 2 (Riemann's Thm)

Suppose that f is bounded and analytic in $0 < |z - z_0| < \varepsilon$ for some $\varepsilon > 0$. Then either f is analytic at z_0 or f has a removable singular point at z_0.

(Pfs = Omitted)

(b) Essential Singular Point

Thm3 (Casorati–Weierstrass Thm)

Suppose that z_0 is an essential singular point of f, and w_0 be any complex number. Then $\forall \varepsilon > 0$, and $\delta > 0$, $\exists\, z \in \{0 < |z - z_0| < \delta\}$ such that

$$|f(z) - w_0| < \varepsilon .$$

Remark: This implies if z_0 is an essential singular point of f, then $\forall\, w_0 \in \mathbb{C}$, $\exists\, z_n \to z_0$ with $z_n \neq z_0$ such that $f(z_n) \to w_0$ as $n \to \infty$.

(Pf: Omitted)

(c) Pole of order m

Thm4: If z_0 is a pole of f, then

$$\lim_{z \to z_0} f(z) = \infty .$$

Pf: Assumption $\Rightarrow$ $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ for some $m \geq 1$

ϕ analytic at z_0 & $\phi(z_0) \neq 0$

$$\Rightarrow \quad \frac{1}{f(z)} = \frac{(z-z_0)^m}{\phi(z)} \to 0 \text{ as } z \to z_0 \quad \#$$

Ch7 Applications of Residues

§85 Evaluation of Improper Integrals

Def: (1) $\int_0^\infty f(x)dx \overset{def}{=} \lim_{R\to\infty} \int_0^R f(x)dx$ (if exists)

(2) $\int_{-\infty}^{\infty} f(x)dx \overset{def}{=\!=} \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)dx$

(if both limits exists)

(3) Cauchy Principal Value (P.V.)

$$P.V. \int_{-\infty}^{\infty} f(x)dx \overset{def}{=\!=} \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \quad \text{(if exists)}$$

Notes: • $\int_{-\infty}^{\infty} f(x)dx$ exists (in the sense of (2))

$$\underset{\not\Longleftarrow}{\Longrightarrow} \quad P.V. \int_{-\infty}^{\infty} f(x)dx \text{ exists} \quad (\text{Ex!})$$

• However, if f is an even function $(f(-x)=f(x))$

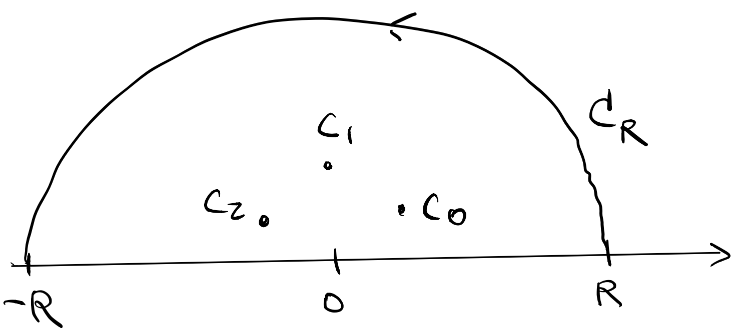
then
$$2\int_0^\infty f(x)dx = \int_{-\infty}^{\infty} f(x)dx = P.V. \int_{-\infty}^{\infty} f(x)dx.$$

§86 Examples

eg1: Evaluate $\int_0^\infty \frac{dx}{x^6+1}$ $\left(=\frac{\pi}{3}\right)$

Soln: Consider $f(z) = \frac{1}{z^6+1}$

Then f is analytic except at the isolated singular points $c_k = e^{i\left(\frac{\pi}{6}+\frac{2k\pi}{6}\right)}$, $k=0,\cdots,5$



Let Γ_R = contour consists of the horizontal line segment from $-R$ to R ($[-R,R]$) and the upper semi-circle C_R of radius R centered at 0.

Then Cauchy Residue's Thm $\Rightarrow$

$$\int_{\Gamma_R} \frac{dz}{z^6+1} = 2\pi i \cdot \sum_{k=0}^{2} \operatorname*{Res}_{z=c_k}\left(\frac{1}{z^6+1}\right)$$

Note that c_k are simple poles of $f(z) = \frac{1}{z^6+1}$

$$\Rightarrow \quad \operatorname*{Res}_{z=c_k}\left(\frac{1}{z^6+1}\right) = \frac{1}{(z^6+1)'\big|_{z=c_k}} = \frac{1}{6c_k^5} = \frac{-c_k}{6}$$

$$\Rightarrow \quad \sum_{k=0}^{2} \operatorname*{Res}_{z=c_k}\left(\frac{1}{z^6+1}\right) = -\frac{c_0}{6} - \frac{c_1}{6} - \frac{c_2}{6} = -\frac{c_0+c_1+c_2}{6}$$

Hence
$$\int_{\Gamma_R} \frac{dz}{z^6+1} = 2\pi i\left(-\frac{1}{6}\right)(c_0+c_1+c_2)$$
$$= \frac{2\pi}{3} \quad (\text{Ex!})$$

On the other hand $\quad |z^6+1| \geq R^6 - 1$ for $z \in C_R$ (and $R > 1$)

$$\therefore \left|\int_{C_R} \frac{dz}{z^6+1}\right| \leq \frac{1}{R^6-1} \cdot \pi R \to 0 \text{ as } R \to \infty.$$

Hence taking $R \to \infty$ in

$$\int_{-R}^{R} \frac{dx}{x^6+1} + \int_{C_R} \frac{dz}{z^6+1} = \int_{\Gamma_R} \frac{dz}{z^6+1} = \frac{2\pi}{3}$$

we have
$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^6+1}=\frac{2\pi}{3}$$

Since $\frac{1}{x^6+1}$ is even, $2\int_0^{\infty}\frac{dx}{x^6+1}=\text{P.V.}\int_{-\infty}^{\infty}\frac{dx}{x^6+1}$
$$=\frac{2\pi}{3}$$
\#\#

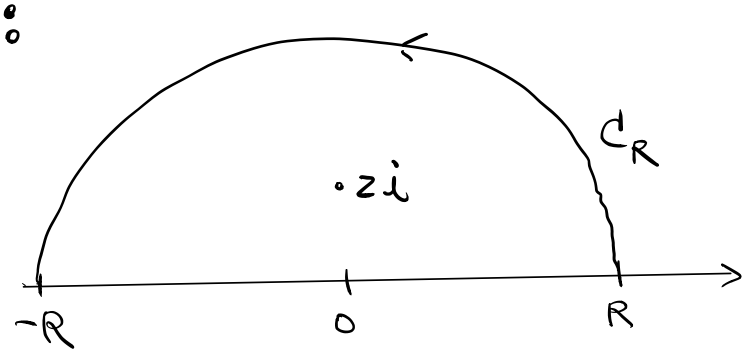
§87 Improper Integrals from Fourier Analysis

To evaluate integrals of the forms
$$\int_{-\infty}^{\infty} f(x)\sin(ax)\,dx \quad \& \quad \int_{-\infty}^{\infty} f(x)\cos(ax)\,dx,$$
consider contour integral $\int_{\Gamma_R} f(z)e^{iaz}\,dz$!

eg Evaluate $\int_0^{\infty}\frac{\cos 2x}{(x^2+4)^2}\,dx$

Soln: Consider contour integral of
$$f(z)e^{i2z}=\frac{e^{i2z}}{(z^2+4)^2}$$

on Γ_R :



Cauchy residue theorem $\Rightarrow$

$$\int_{-R}^{R} \frac{e^{i2x}}{(x^2+4)^2}\,dx + \int_{C_R} \frac{e^{i2z}}{(z^2+4)^2}\,dz = 2\pi i \operatorname*{Res}_{z=2i} \frac{e^{i2z}}{(z^2+4)^2}$$

(pole of order 2)

$$= 2\pi i \left(-\frac{5e^{-4}}{32} i \right) \quad (Ex!)$$

$$\Rightarrow \int_{-R}^{R} \frac{\cos 2x + i\sin 2x}{(x^2+4)^2}\,dx + \int_{C_R} \frac{e^{i2z}}{(z^2+4)^2}\,dz = \frac{5e^{-4}}{16}\pi$$

taking real part

$$\int_{-R}^{R} \frac{\cos 2x}{(x^2+4)^2}\,dx + \operatorname{Re}\int_{C_R} \frac{e^{i2z}}{(z^2+4)^2}\,dz = \frac{5e^{-4}}{16}\pi$$

Note that on the semicircle C_R $(z = Re^{i\theta},\ 0 \le \theta \le \pi)$

$$|e^{i2z}| = |e^{i2(R\cos\theta + iR\sin\theta)}|$$

$$= |e^{-2R\sin\theta + i2R\cos\theta}| = e^{-2R\sin\theta} \le 1$$

Together with $|z^2+4| \ge R^2 - 4$ (for $R > 2$),

we have

$$\left| \mathrm{Re} \int_{C_R'} \frac{e^{i2z}}{(z^2+4)^2} dz \right| \le \left| \int_{C_R'} \frac{e^{i2z} dz}{(z^2+4)^2} \right| \le \frac{1}{(R^2-4)^2} \cdot \pi R$$

$$\to 0 \quad \text{as } R \to \infty$$

$\therefore$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos 2x}{(x^2+4)^2} dx = \frac{5e^{-4}}{16}\pi$$

$$\Rightarrow \quad \int_0^\infty \frac{\cos 2x}{(x^2+4)^2} dx = \frac{5e^{-4}}{32}\pi \quad \#$$

Remark: A crucial point in the above method is

$$\lim_{R\to\infty} \left| \int_{C_R'} f(z) e^{iaz} dz \right| = 0.$$

In our eg. $|f(z)| \le \frac{M}{R^4}$ on C_R' (for R large)

$$\Rightarrow \quad \left| \int_{C_R'} f(z) e^{iaz} dz \right| \le \frac{M}{R^4} \cdot \pi R \to 0.$$

It is easy to see that the method works for

$$|f(z)| \leq \frac{M}{R^{1+\delta}} \quad \text{for some } \delta > 0$$

However, we do need to handle cases like

$$\int_0^\infty \frac{x \sin 2x}{x^2+3}\,dx,$$

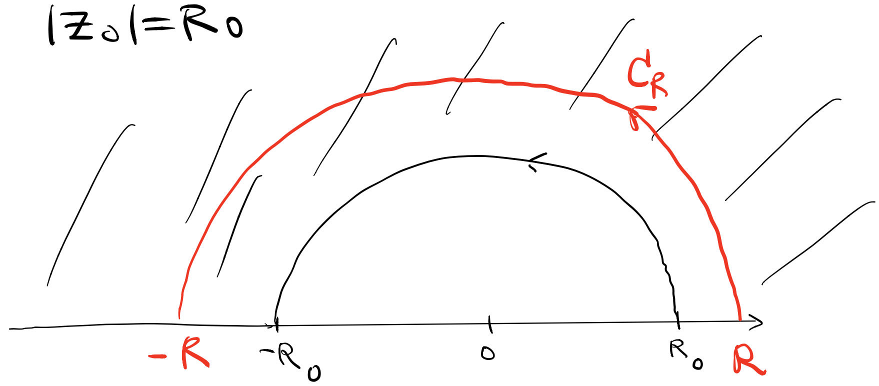
where $|f(z)| \leq \frac{M}{R}$ only.

§88 Jordan's Lemma

Thm (Jordan's Lemma)

Suppose that

(a) a function $f(z)$ analytic at all points in the upper half plane $y \geq 0$ that are exterior to a circle $|z_0| = R_0$



(b) C_R = positively oriented semicircle of radius $R > R_0$

$= \{ z = Re^{i\theta},\ 0 \le \theta \le \pi \}$

(c) there is a constant $M_R > 0$ such that

$$f(z) \le M_R, \quad \forall z \in C_R$$

and $\lim_{R \to \infty} M_R = 0$.

Then $\forall a > 0$,

$$\lim_{R \to \infty} \int_{C_R} f(z) e^{iaz} dz = 0.$$

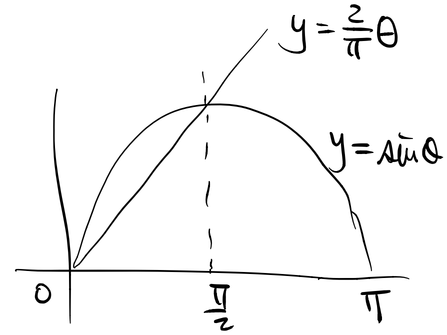
Pf of Jordan's Lemma needs:

Lemma (Jordan Inequality)

$$\boxed{\int_0^{\pi} e^{-R\sin\theta} d\theta < \frac{\pi}{R} \qquad (R > 0)}$$

Pf: By property of $\sin\theta$, we have

$$\frac{2}{\pi}\theta \le \sin\theta, \quad \forall \theta \in [0, \tfrac{\pi}{2}]$$



$\Rightarrow \forall R>0$,

$$e^{-R\sin\theta} \le e^{-\frac{2R}{\pi}\theta}, \quad \forall \theta \in [0, \tfrac{\pi}{2}]$$

$$\Rightarrow \quad \int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta \le \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta} d\theta = \frac{\pi}{2R}(1-e^{-R})$$

Then by $\displaystyle\int_{\frac{\pi}{2}}^{\pi} e^{-R\sin\theta} d\theta = \int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta$ (EX!),

we have $$\int_0^{\pi} e^{-R\sin\theta} d\theta < \frac{\pi}{R}. \quad \#$$

Pf of Jordan's Lemma:

Note that $$|e^{iaz}| = |e^{ia(R\cos\theta + iR\sin\theta)}| = e^{-aR\sin\theta}$$

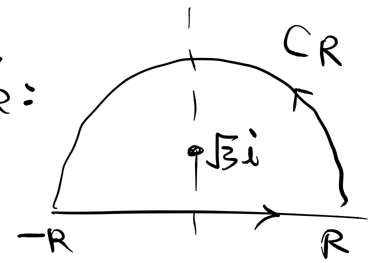
$$\Rightarrow \left|\int_{C_R} f(z) e^{iaz} dz\right| \le M_R \int_0^{\pi} |e^{iaz}| R \, d\theta$$

$$= M_R \cdot R \cdot \int_0^{\pi} e^{-aR\sin\theta} d\theta$$

$$< M_R \cdot R \cdot \frac{\pi}{aR} = \frac{\pi}{a} M_R \longrightarrow 0 \text{ as } R \to \infty$$

eg: Evaluate $\int_0^\infty \frac{x \sin 2x}{x^2+3}\,dx$

Soln: $f(z)e^{i2z} = \frac{z}{z^2+3} e^{i2z}$ on Γ_R:



Note that $\operatorname{Res}_{z=\sqrt{3}i} \left(\frac{z e^{i2z}}{z^2+3} \right) = \frac{1}{2} e^{-2\sqrt{3}}$ (Ex!)
(simple pole)

Cauchy Residue Thm $\Rightarrow$

$$\int_{-R}^{R} \frac{x e^{i2x}}{x^2+3}\,dx + \int_{C_R} \frac{z}{z^2+3} e^{i2z}\,dz = 2\pi i \left(\frac{1}{2} e^{-2\sqrt{3}}\right)$$

Since $\left|\frac{z}{z^2+3}\right| \le \frac{R}{R^2-3}$ on C_R

$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{M_R} \to 0$ as $R \to \infty$.

Jordan's Lemma $\Rightarrow$ $\lim_{R\to\infty} \int_{C_R} \frac{z}{z^2+3} e^{i2z}\,dz = 0$

Hence

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x(\cos 2x + i \sin 2x)}{x^2+3}\,dx = \pi e^{-2\sqrt{3}} i$$

Imaginary part $\Rightarrow$ $\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin 2x\,dx}{x^2+3} = \pi e^{-2\sqrt{3}}$

$$\therefore \quad \int_0^\infty \frac{x\sin 2x\,dx}{x^2+3} = \frac{\pi e^{-2\sqrt{3}}}{2} \quad \#$$