Thm3 Suppose that a function f is analytic inside and on a positively oriented circle $C_R = \{ |z - z_0| = R \}$ $(R > 0)$. Let $M_R = \max\{|f(z)| : z \in C_R\}$, then

$$\boxed{|f^{(n)}(z_0)| \le \frac{n!\, M_R}{R^n} \qquad (\forall\, n = 1, 2, 3, \cdots)}$$

(Cauchy Inequality.)

Pf: By Cauchy Integral Formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)\,dz}{(z - z_0)^{n+1}}, \quad n = 1, 2, \cdots$$

Note that on C_R: $|z - z_0| = R$ and length of $C_R = 2\pi R$, we have

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{n!\, M_R}{R^n}$$

#

§58 Liouville's Theorem and the Foundamental Theorem of Algebra

Thm 1 (Liouville's Thm) If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the complex plane.

Pf: Let M be the bound of f, i.e.

$$|f(z)| \leq M, \quad \forall z \in \mathbb{C}.$$

Then $\forall z_0 \in \mathbb{C}$ and any $R > 0$,

f is entire $\Rightarrow$ f analytic inside and on $C_R = \{|z - z_0| = R\}$

(Cauchy Inequality for $n=1$) $\Longrightarrow$ $|f'(z_0)| \leq \frac{M_R}{R} \leq \frac{M}{R}$.

(where $M_R = \max\{|f(z)| : z \in C_R\} \leq M$.)

Letting $R \to +\infty$, we have $f'(z_0) = 0$

$\Rightarrow$ $f' \equiv 0$ on $\mathbb{C}$

$\Rightarrow \quad f \equiv \text{constant on } \mathbb{C}$ ※

Thm 2 (Foundamental Theorem of Algebra)

Any polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $(a_n \neq 0)$ of degree n $(n \geq 1)$ has at least one zero.

(i.e. $\exists\, z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.)

Pf: Suppose not. Then $P(z) \neq 0$, $\forall z \in \mathbb{C}$.

$$\Rightarrow \quad f(z) = \frac{1}{P(z)} \text{ is an entire function.}$$

To prove f is bounded, we rewrite

for $z \neq 0$,

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

$$= z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_k}{z^{n-k}} + \cdots + \frac{a_0}{z^n} \right)$$

Let $R = \max_{k=0,1,\cdots,n-1} \left(\sqrt[n-k]{\frac{2n|a_k|}{|a_n|}} \right) > 0$ (unless $a_0 = \cdots = a_{n-1} = 0$ $\Rightarrow P(z) = a_n z^n$ has zero)

Then for $|z| > R$,

$$\left|\frac{a_k}{z^{n-k}}\right| = \frac{|a_k|}{|z|^{n-k}} < \frac{|a_k|}{R^{n-k}} \leq \frac{|a_k|}{\left(\frac{2n|a_k|}{|a_n|}\right)} = \frac{|a_n|}{2n}$$

$\forall k = 0, 1, \cdots, n-1.$

$$\Rightarrow \left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \leq n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2}, \quad \forall |z| > R$$

Hence for $|z| > R$,

$$\begin{aligned} |P(z)| &= |z^n| \left| a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right) \right| \\ &\geq |z|^n \left[|a_n| - \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \right] \\ &\geq |z|^n \left(|a_n| - \frac{|a_n|}{2} \right) \\ &\geq \frac{|a_n|}{2} R^n \end{aligned}$$

$$\Rightarrow |f(z)| = \left|\frac{1}{P(z)}\right| \leq \frac{2}{|a_n| R^n}, \quad \forall |z| > R$$

Note that $P(z) \neq 0$, $\forall z \in \mathbb{C} \Rightarrow f(z) = \frac{1}{P(z)}$ is a continuous function on $\overline{B_R(0)} = \{ |z| \leq R \}$ (closed and bounded set in $\mathbb{C}$)

$\Rightarrow \exists\, M_1 > 0$ such that

$$|f(z)| \leq M_1, \quad \forall z \in \overline{B_R(0)}.$$

Hence
$$|f(z)| \leq \max\left\{ M_1, \frac{2}{|a_n| R^n} \right\}, \quad \forall z \in \mathbb{C}$$

i.e. $f(z) = \frac{1}{P(z)}$ is a bounded entire function

By Liouville's Thm, $f(z) = \frac{1}{P(z)} = \text{const}.$

$\Rightarrow$ degree $n = 0$ which is a contradiction (as $n \geq 1$)

$\therefore$ $P(z)$ has at least a zero. ※

<u>Note</u>: By fundamental thm of algebra, we immediately have
$$P(z) = a_n (z - z_1) \cdots (z - z_n)$$
where $z_1, \cdots, z_n$ are zeros of $P(z)$ (may not distinct.)

§59 Maximum Modulus Principle

Lemma (Gauss' mean value theorem)

If $f(z)$ is analytic inside and on $C_\rho = \{|z-z_0|=\rho\}$

then
$$\boxed{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta}$$

Pf: Cauchy Integral Formula $(n=0)$

$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)\,dz}{z-z_0}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})\,\rho i e^{i\theta}\,d\theta}{\rho e^{i\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta \; . \#$$

Lemma: Suppose that f is analytic in $\{|z-z_0|<\rho_0\}$

and $|f(z)| \le |f(z_0)|$, $\forall z \in \{|z-z_0|<\rho_0\}$.

Then $f(z) \equiv f(z_0)$, $\forall z \in \{|z-z_0|<\rho_0\}$.

Pf : $\forall\ 0<\rho<\rho_0$, Gauss' mean value theorem

$$\Rightarrow \quad f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta$$

$$\Rightarrow \quad |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\,d\theta$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta = |f(z_0)|$$

$$\Rightarrow \quad \int_0^{2\pi} \left[|f(z_0)| - |f(z_0+\rho e^{i\theta})| \right] d\theta = 0$$

Since $|f(z_0)| \ge |f(z_0+\rho e^{i\theta})|$ $\quad \forall\ 0<\rho<\rho_0$, $0\le\theta\le 2\pi$,

we have $|f(z_0)| - |f(z_0+\rho e^{i\theta})| \ge 0$ & continuous on $0\le\theta\le 2\pi$

$$\Rightarrow \quad |f(z_0)| = |f(z_0+\rho e^{i\theta})|, \quad \forall\ 0<\rho<\rho_0,\ 0\le\theta\le 2\pi$$

i.e. $|f(z_0)| = |f(z)|$, $\forall\ z \in \{|z-z_0|<\rho_0\}$

Recall that an analytic function with constant modulus is in fact constant (eg4 of §26) $\Rightarrow f(z) = f(z_0)$

※

Thm (Maximum Modulus Principle)

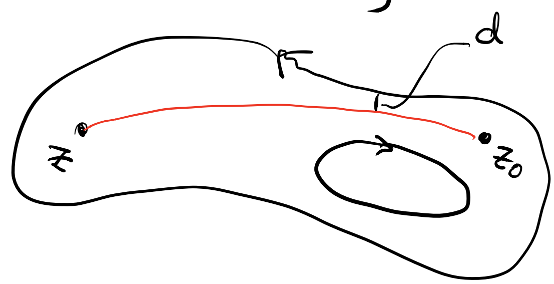
If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D, i.e. $\exists$ no point $z_0 \in D$ such that $|f(z)| \le |f(z_0)|$, $\forall z \in D$.

Note: This is equivalent to: If f is analytic in a domain D and $\exists\, z_0 \in D$ such that $|f(z)| \le |f(z_0)|$, $\forall z \in D$. Then f is a constant function.

Pf: Suppose $\exists\, z_0 \in D$ such that

$$|f(z)| \le |f(z_0)|, \quad \forall z \in D.$$

Then $\forall z \in D$, connect z_0 to z by a contour L in D.

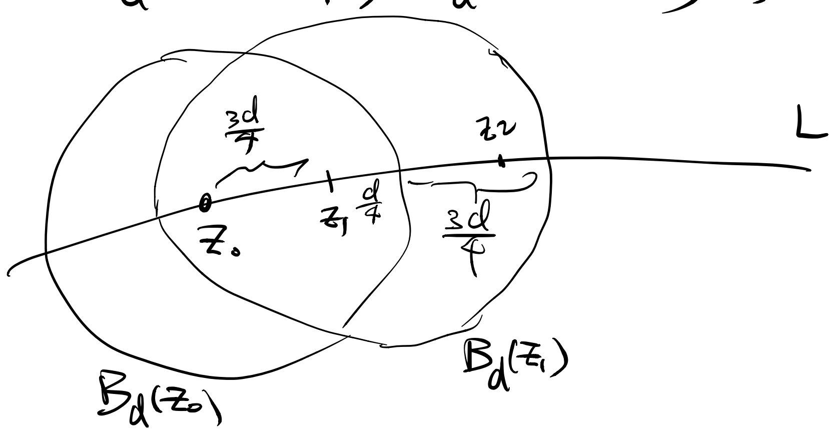


Let d = distance from L to ∂D.

Then (by compactness of L) there exists finitely many points $z_1, \cdots, z_{n-1}$ such that

$$B_d(z_0) \cup B_d(z_1) \cup \cdots \cup B_d(z_{n-1}) \supset L$$

and $B_d(z_0) \ni z_1$, $B_d(z_1) \ni z_2$, $\cdots$, $B_d(z_{n-1}) \ni z$



Then applying the lemma to $B_d(z_0)$

$\Rightarrow \quad f(z_1) = f(z_0)$

$\Rightarrow \quad |f(z_1)| \geq |f(s)|, \ \forall s \in B_d(z_1)$

$\Rightarrow \quad f(z_2) = f(z_1) = f(z_0)$

$\vdots$ and so on

we have $f(z) = f(z_{n-1}) = \cdots = f(z_1) = f(z_0)$.

#

Cor: Suppose that a function f is continuous on a closed and bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reach, occurs somewhere on the boundary ∂R of R and never in the interior.

(Pf: Immediately from maximum (modulus) principle.)

Note: The corollary holds for real part and imaginary part of an non-constant analytic function as

f analytic, non-const. $\Rightarrow$ $g = e^f$ analytic, nonconst

& $|g| = e^{\mathrm{Re} f}$.

(For Imaginary part $|e^{if}| = e^{\mathrm{Im} f}$.)

Ch5 Series

§60 Convergence of Sequences

Def: (i) An infinite sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers has a limit z if

$\forall \varepsilon > 0$, $\exists$ positive integer n_0 such that

$$|z_n - z| < \varepsilon, \quad \forall n > n_0.$$

(ii) When limit z exists, the sequence is said to be converge to z and denoted by $\lim_{n\to\infty} z_n = z$.

(iii) If a sequence has no limit, it diverges.

Fact: If limit exists, it is unique.

(Pf: Ex!)

Thm: Suppose that $z_n = x_n + iy_n$ $(n=1,2,3,\cdots)$ &

$z = x + iy$

Then $\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

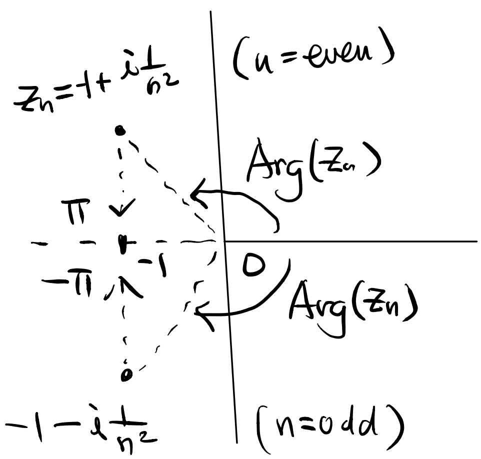
i.e. $$\boxed{\lim_{n\to\infty}(x_n + iy_n) = \lim_{n\to\infty} x_n + i\lim_{n\to\infty} y_n}$$

And hence $$\boxed{\lim_{n\to\infty} |z_n| = \left| \lim_{n\to\infty} z_n \right|}$$

eg However $\lim_{n\to\infty} \operatorname{Arg} z_n \neq \operatorname{Arg}\left(\lim_{n\to\infty} z_n\right)$ in general.

$$\lim_{n\to\infty} \underbrace{\left(-1 + i\frac{(-1)^n}{n^2}\right)}_{= z_n} = -1$$

Principal argument of z_n
$= \operatorname{Arg}(z_n)$



$\therefore \lim_{n\to\infty} \operatorname{Arg}\left(-1 + i\frac{(-1)^n}{n^2}\right)$ doesn't exist

$\left(\begin{array}{l} \text{even terms} \to \pi \\ \text{odd terms} \to -\pi \end{array}\right)$

<u>In summary</u> If $z_n \to z$, then

$$\begin{cases} \operatorname{Re} z_n \to \operatorname{Re} z, \\ \operatorname{Im} z_n \to \operatorname{Im} z, \\ |z_n| \to |z| \end{cases}$$

But $\operatorname{Arg} z_n$ may <u>not</u> even converge!

§61 <u>Convergence of Series</u>

<u>Def</u> (i) An infinite <u>series</u> $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$

of complex numbers <u>converges</u> to the <u>sum</u> S if

the <u>sequence of partial sum</u>

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N, \quad N = 1, 2, 3 \cdots$$

<u>converges</u> to S, ie. $\lim_{N\to\infty} S_N = S$

$$\left(\lim_{N\to\infty} \sum_{n=1}^{N} z_n = S \right)$$

We then write $\sum_{n=1}^{\infty} z_n = S$.

(ii) When a series doesn't converge, we say that it is <u>diverges</u>.

Clearly, we have

<u>Thm</u>: Suppose that $z_n = x_n + i y_n$ $(n = 1, 2, 3, \cdots)$
and $S = X + iY$.

Then

$$\sum_{n=1}^{\infty} z_n = S \iff \sum_{n=1}^{\infty} x_n = X \ \& \ \sum_{n=1}^{\infty} y_n = Y.$$

i.e. $$\boxed{\sum_{n=1}^{\infty} (x_n + i y_n) = \left(\sum_{n=1}^{\infty} x_n\right) + i \left(\sum_{n=1}^{\infty} y_n\right)}$$

Cor 1: If a series of complex numbers converges, then the n-th term converges to zero as n tends to infinity.

(Pf: Ex! using the above and the corresponding result in $\mathbb{R}$.)