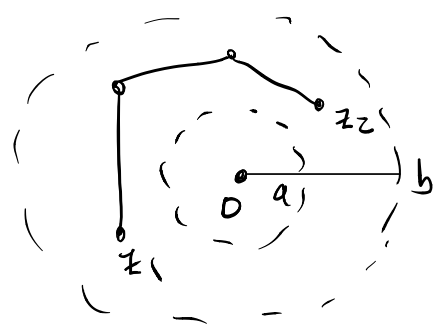
§12 Regions in the complex plane

Def: (1) $B_\varepsilon(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \varepsilon \}$ is called the <u>ε-neighborhood</u> (ε-nbd) of the point z_0

(2) $B_\varepsilon(z_0) \setminus \{z_0\} = \{ z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon \}$ is called the <u>deleted ε-neighborhood</u> of z_0. (or delected ε-nbd)

Def: S is <u>connected</u> if $\forall z_1, z_2 \in S$, $\exists$ a <u>polygonal line</u> in S joining z_1 & z_2.

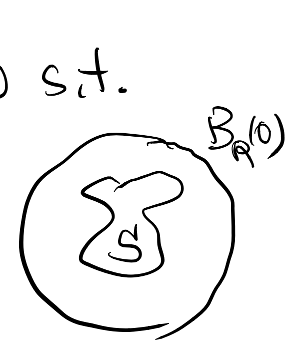
eg: <u>Annulus</u> $\{ a < |z| < b \}$ $(0 < a < b < +\infty)$ is connected



Def: A set S is <u>bounded</u>, if $\exists R > 0$ s.t.

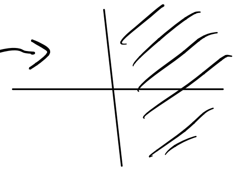
$$S \subset B_R(0)$$

(i.e. $|z| < R$, $\forall z \in S$)



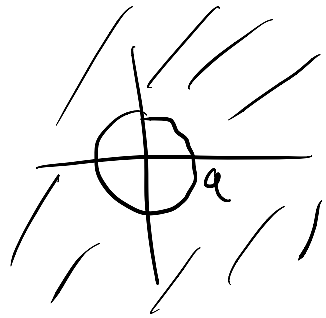
eg: $B_\varepsilon(z_0)$, $\{a<|z|<b\}$ $(b<+\infty)$ are bounded

$\{\mathrm{Re}\, z>0\}$



$\{\mathrm{Re}\, z>0\}$

& $\{a<|z|<+\infty\}$



are <u>unbounded</u>.

Ch2 Analytic Functions

§13 Functions & Mappings

Let S be a set of cpx numbers.

Def: (1) A function f defined on S is a rule that assigns to each $z \in S$, a complex number w.

Notation: $w = f(z) \in \mathbb{C}$.

(Sometime, f is called complex-valued function if needed)

(2) The cpx number $w = f(z)$ is the called the value of f at z.

(3) S is called the domain (of definition) of f.

Convention: when the domain of f is not mentioned, we agree that the largest possible set is to be taken.

Note: If $z = x+iy$ and $w = f(z) = u+iv$

i.e. $u+iv = f(z) = f(x+iy)$

$\Rightarrow$ u, v the real and imaginary parts of f are real-valued functions of 2-variables

$u = u(x,y)$ & $v = v(x,y)$

And $f(z) = u(x,y) + i v(x,y)$

eg: $f(z) = z^2$, then

$u+iv = f(x+iy) = (x+iy)^2 = (x^2-y^2) + 2ixy$

i.e. $\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$

Convention: If $f = u+iv$ with $v \equiv 0$, then f is a real-valued function of a cpx variable.

eg: $f(z) = |z|^2 = x^2+y^2$

$\Leftrightarrow \begin{cases} u = x^2+y^2 \\ v = 0 \end{cases}$

Terminology

(1) $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $a_n \neq 0$, $a_0, \cdots, a_n \in \mathbb{C}$

is a polynomial of degree n.

(2) Quotient $\frac{P(z)}{Q(z)}$ of polynomials $P(z)$ & $Q(z)$ are called rational functions (defined at z with $Q(z) \neq 0$).

Polar form:

Using polar coordinates or exponential forms of z

$$\begin{cases} u = u(r,\theta) \\ v = v(r,\theta) \end{cases}$$

i.e. $$f(z) = u(r,\theta) + i v(r,\theta) \quad \text{for } z = re^{i\theta}.$$

eg: $w = z^2$ with $z = re^{i\theta}$

$$\Rightarrow \quad w = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2\cos 2\theta + ir^2 \sin 2\theta$$

i.e. $$\begin{cases} u = r^2 \cos 2\theta \\ v = r^2 \sin 2\theta \end{cases}$$

Multiple-valued functions: assigns more than one value to a point z in the domain of definition.

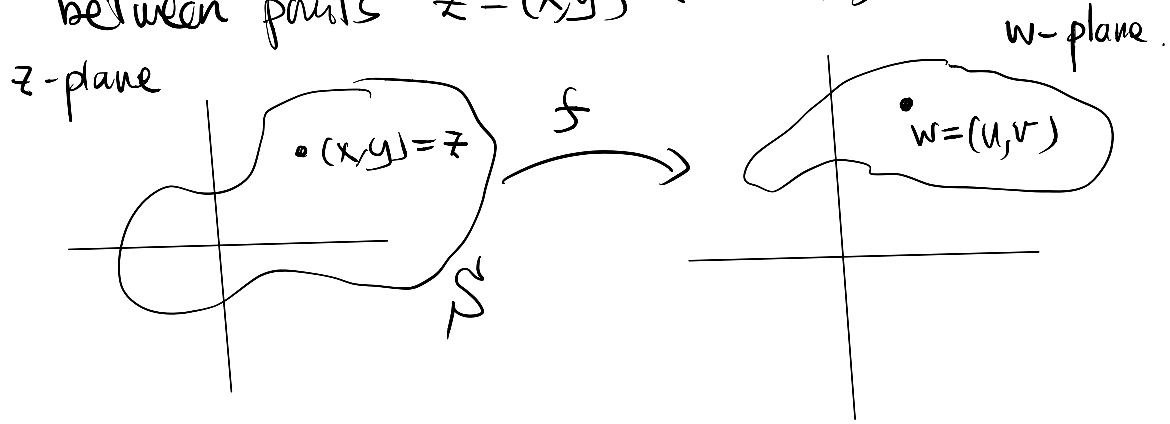
eg: $z \mapsto z^{\frac{1}{n}} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$, $k = 0, 1, \cdots, n-1$

is a multiple-valued function for $n \geq 2$.

Terminology

(1) Mapping or transformation

when a function f is thought of correspondence between points $z = (x, y)$ & $w = (u, v)$.



(2) The point $w = (u, v)$ is called the image of the point $z = (x, y)$ under the mapping (transformation) $w = f(z)$.

(3) Range of $f = \{ w : w = f(z), \forall z \in S \}$

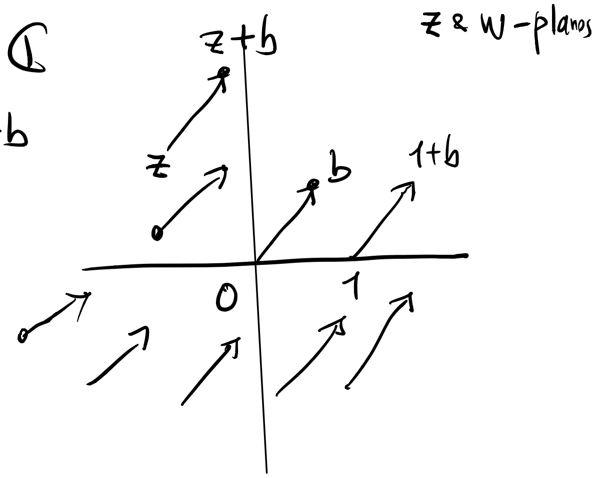
(4) Inverse image of a point w_0

$$f^{-1}(w_0) \overset{def}{=} \{ z \in S : f(z) = w_0 \}$$

Egs: (1) For any fixed $b \in \mathbb{C}$

$$w = f(z) = z + b$$

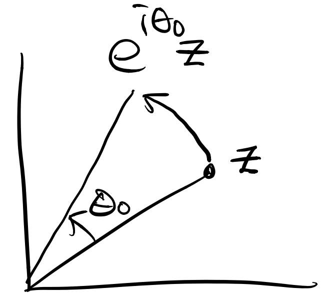
is a <u>translation</u>



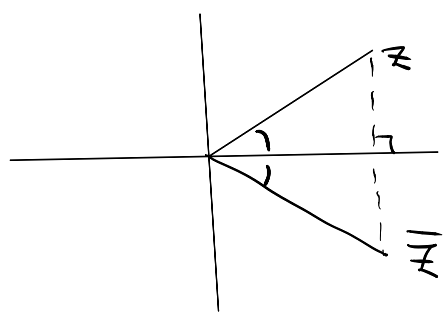
(2) For any fixed $\theta_0 \in \mathbb{R}$

$$w = f(z) = e^{i\theta_0} z$$

is a <u>rotation</u> by angle θ_0 in counterclockwise direction



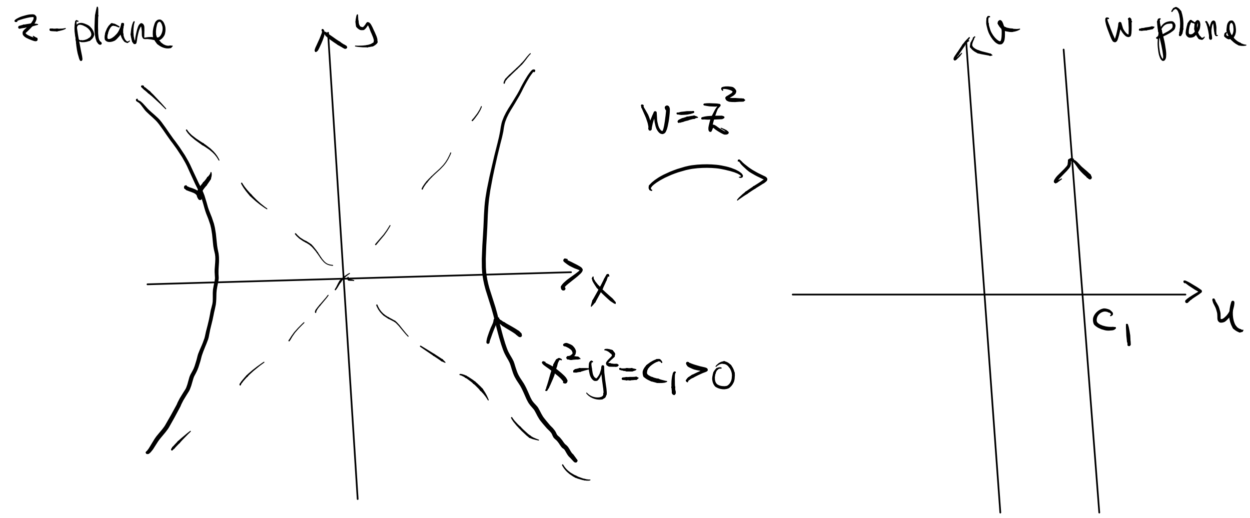
(3) The function $w = f(z) = \overline{z}$ is a <u>reflection</u> in x-axis



§14 <u>The mapping $w = z^2$</u>

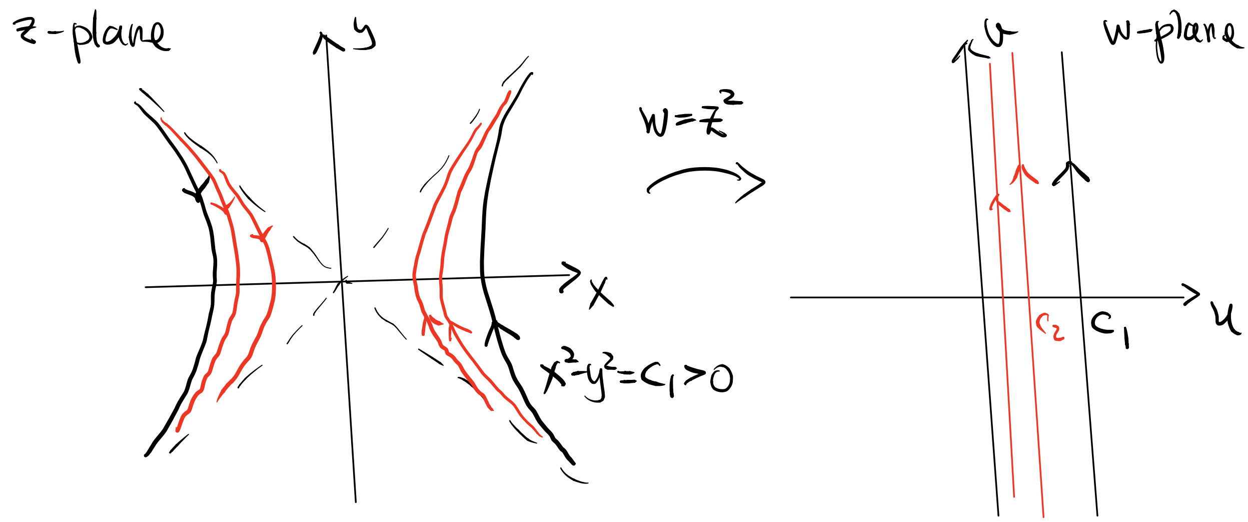
The $w = z^2$ can be thought of as the transformation

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$



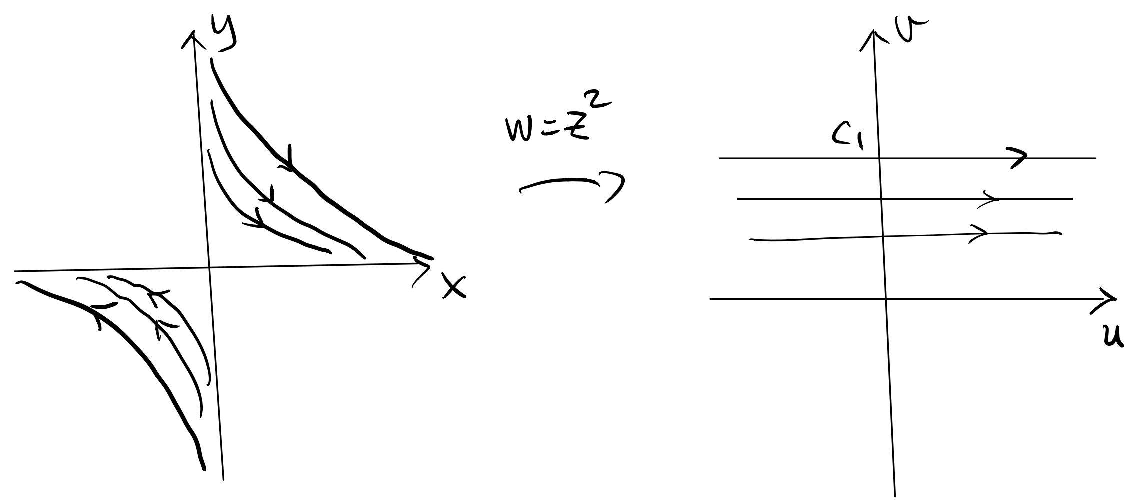
Consider

$$\begin{cases} u = x^2-y^2 = c_1 \\ v = 2xy \text{ arbitrary} \end{cases}$$

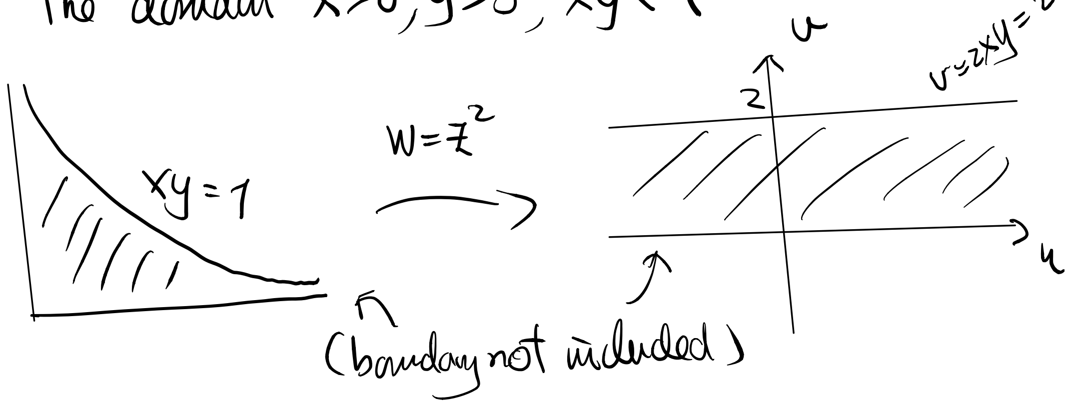


Ex: What happen for $c_1 < 0$ ($c_1 = 0$)?

Similarly, we can consider $v = 2xy = c_1$ ($c_1 > 0$)



eg1: The domain $x>0, y>0, xy<1$



will be mapped to the horizontal strip $0<v<2$ under $w=z^2$.

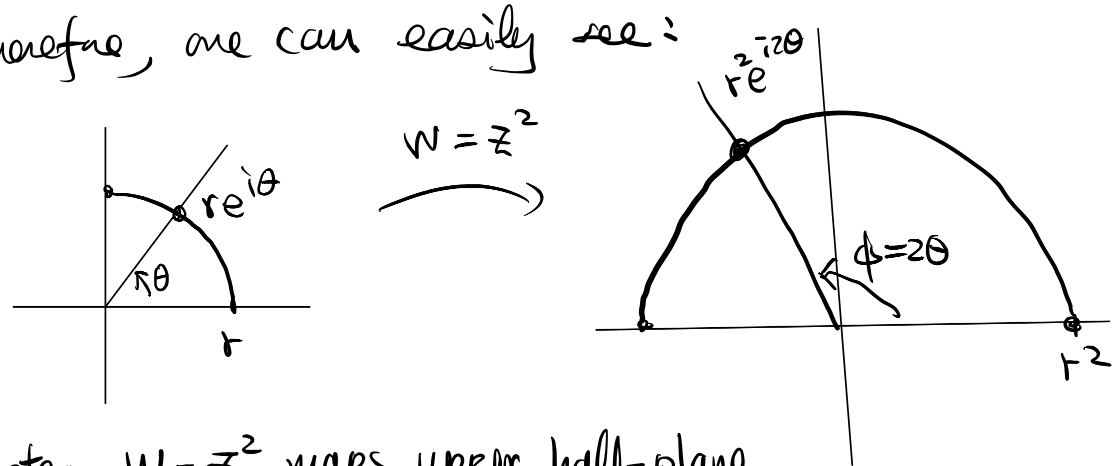
eg2: In exponential form for $w=z^2$

Let $z=re^{i\theta}$ and $w=\rho e^{i\phi}$.

Then $$\rho e^{i\phi}=(re^{i\theta})^2=r^2e^{i2\theta}$$

$$\begin{cases}\rho=r^2\\ \phi=2\theta\end{cases}$$ this is the polar form of the transformation.

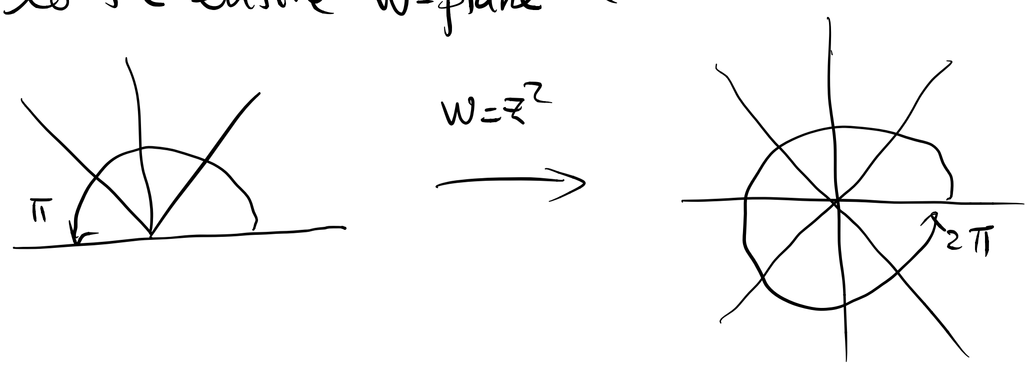
Therefore, one can easily see:



Note: $W = z^2$ maps upper half-plane

$\{ r \geq 0,\ 0 \leq \theta \leq \pi \}$ (with boundary)

to the entire W-plane (but not 1-1)



§15 Limits

Def: The function $f(z)$ has a limit w_0 as z approaches z_0, denoted by

$$\lim_{z \to z_0} f(z) = w_0$$

means $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that

$$|f(z) - w_0| < \varepsilon, \quad \forall\ 0 < |z - z_0| < \delta.$$

Thm If $\lim_{z \to z_0} f(z)$ exists, it is unique.

§16 Theorems on Limits

Thm 1: Suppose that $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$

& $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$

Then

$$\begin{cases} \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 \end{cases} \iff \lim_{z\to z_0} f(z) = w_0 .$$

eg: (i) $\lim_{z\to 1}\left(i\frac{\bar{z}}{2}\right) = \lim_{(x,y)\to(1,0)}\left(i\cdot\frac{\overline{x+iy}}{2}\right)$

$$= \lim_{(x,y)\to(1,0)}\left(\frac{y}{2} + i\frac{x}{2}\right) = \frac{i}{2}.$$

(ii) $\lim_{z\to 0}\frac{z}{\bar{z}} = \lim_{z\to 0}\frac{re^{i\theta}}{re^{-i\theta}} = \lim_{z\to 0} e^{2i\theta}$

$$= \lim_{(x,y)\to(0,0)}\left(\cos 2\theta + i\sin 2\theta\right)$$

limit doesn't exists

different values in different directions.

<u>Thm2</u>: Suppose that $\lim_{z\to z_0} f(z) = w_0$, $\lim_{z\to z_0} F(z) = W_0$

then

(1) $$\lim_{z\to z_0}[f(z)\pm F(z)] = w_0 + W_0$$

$$\left(\text{i.e. } \lim_{z\to z_0}[f(z)\pm F(z)] = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} F(z)\right)$$

(2) $$\lim_{z\to z_0} f(z)F(z) = w_0 W_0$$

$$\left(\text{i.e. } \lim_{z\to z_0} f(z)F(z) = \lim_{z\to z_0} f(z) \lim_{z\to z_0} F(z)\right)$$

(3) if $W_0 \neq 0$, $$\lim_{z\to z_0}\frac{f(z)}{F(z)} = \frac{w_0}{W_0}$$

$$\left(\text{i.e. } \lim_{z\to z_0}\frac{f(z)}{F(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} F(z)} \quad \text{provided } \lim_{z\to z_0} F(z)\neq 0\right)$$

§17 Limits involving the "point at infinity"

<u>Def</u>: The <u>extended complex plane</u> is the union of complex plane $\mathbb{C}$ (= the set of cpx numbers) and the <u>point of infinity</u> $\{\infty\}$.

Thm (Def) If z_0 & w_0 are points in the z & w-plane respectively, then

(1) $$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \left(\frac{1}{f(z)} \right) = 0$$

(2) $$\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$

Moreover

(3) $$\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$$