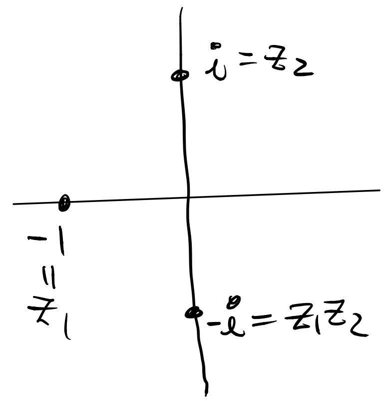
§9 Arguments of Products and Quotients

eg: let $z_1 = -1$, $z_2 = i$



$$\Rightarrow \begin{cases} z_1 = e^{i\,\mathrm{Arg} z_1} = e^{i\pi} \\ z_2 = e^{i\,\mathrm{Arg} z_2} = e^{i\frac{\pi}{2}} \end{cases}$$

& $z_1 z_2 = -i$

$$= e^{i\,\mathrm{Arg}(z_1 z_2)} = e^{i\left(-\frac{\pi}{2}\right)}$$

$$\mathrm{Arg}(z_1 z_2) = -\frac{\pi}{2} \neq \mathrm{Arg} z_1 + \mathrm{Arg} z_2 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$

So the formula $e^{i(\theta_1+\theta_2)} = e^{i\theta_1} e^{i\theta_2}$

$$\not\Rightarrow \mathrm{Arg}(z_1 z_2) = \mathrm{Arg} z_1 + \mathrm{Arg} z_2$$

only $\Rightarrow$ $\boxed{\arg(z_1 z_2) = \arg z_1 + \arg z_2}$

(For subsets S_1, S_2 of $\mathbb{R}$,
$S_1 + S_2 \overset{\text{def}}{=} \{a+b : a \in S_1, b \in S_2\}$)

That is, the formula is good for set $\arg z$,
but not good for principal value $\mathrm{Arg} z$.

§10 Roots of complex Numbers

Note: $r_1 e^{i\theta_1} = r_2 e^{i\theta_2} \iff \begin{cases} r_1 = r_2 \\ \theta_1 = \theta_2 + 2k\pi, \ k \in \mathbb{Z} \end{cases}$

Then for $z_0 = r_0 e^{i\theta_0} (\neq 0)$

$$\boxed{\begin{aligned} c_k &= \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, 2, \cdots, n-1 \\ &= \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \\ &\text{are all the distinct } n\text{-root of } z_0 \end{aligned}}$$

i.e $\begin{cases} \bullet \ c_k^n = z_0 \quad \forall k = 0, 1, 2, \cdots n-1 \\ \bullet \ \text{if } w^n = z_0, \text{ then } w = c_k \text{ for some } k = 0, 1, \cdots, n-1 \end{cases}$

Notations:

(1) $z_0^{\frac{1}{n}}$ = set of all n-roots of z_0

$= \{c_0, c_1, \cdots, c_{n-1}\}$

$= \left\{ c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)} : k = 0, 1, \cdots, n-1 \right\}$

In this notation,

$$r_0^{\frac{1}{n}} = \left\{ c_k = \sqrt[n]{r_0}\, e^{i\frac{2k\pi}{n}} : k=0,1,\cdots,n-1 \right\}$$

but $\sqrt[n]{r_0}$ = the positive n-root of r_0 (as in real case)

which is a positive real number.

(2) <u>Principal n-root</u>

If $z_0 = r_0 e^{i\theta_0}$ with $\theta_0 = \mathrm{Arg}\, z_0 \in (-\pi, \pi]$,

then $$c_0 = \sqrt[n]{r_0}\, e^{i\frac{\mathrm{Arg}\, z_0}{n}} \quad \left(= \sqrt[n]{r_0}\, e^{i\frac{\theta_0}{n}}\right)$$

(i.e. $k=0$ in the above formula)

is called the <u>Principal n-roots</u> of z_0

(3) Let $\omega_n = e^{i\frac{2\pi}{n}}$. Then

$$\begin{cases} \omega_n^k = e^{i\frac{2k\pi}{n}}, & k=0,1,\cdots,n-1 \\ \omega_n^n = e^{i2\pi} = 1 \end{cases}$$

Hence for $z_0 = r_0 e^{i\mathrm{Arg}\, z_0}$, then

$$z_0^{\frac{1}{n}} = \sqrt[n]{r_0}\, e^{i\left(\frac{\mathrm{Arg}\, z_0}{n} + \frac{2k\pi}{n}\right)} = \left(\sqrt[n]{r_0}\, e^{i\frac{\mathrm{Arg}\, z_0}{n}}\right) e^{i\frac{2k\pi}{n}}$$

$$= c_0\, \omega_n^k, \quad k=0,1,\cdots,n-1$$

ω_n is called the <u>n-root of unity</u>.

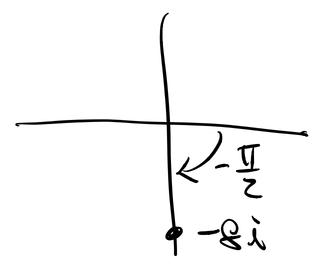
And $\forall z_0$ with principal n-root c_0, we have

$$z_0^{\frac{1}{n}} = \{ c_0 \omega_n^k : k = 0, 1, \cdots, n-1 \}$$

$$= \{ \text{"principal n-root of } z_0\text{"} \times (\text{n-root of unity})^k : k = 0, 1, \cdots, n-1 \}$$

§11 <u>Examples</u>

<u>eg1</u>: Find $(-8i)^{1/3}$



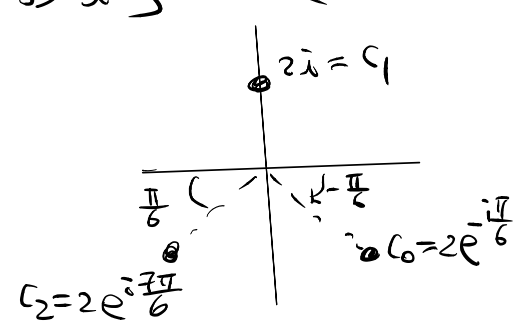
<u>Soln</u>: $-8i = 8e^{i(-\frac{\pi}{2})}$

$$\Rightarrow (-8i)^{1/3} = \sqrt[3]{8}\, e^{i\left(\frac{-\frac{\pi}{2}}{3} + \frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2$$

$$= 2e^{i\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2$$

$$= \{ 2e^{-\frac{i\pi}{6}}, 2e^{i\frac{\pi}{2}}, 2e^{i\frac{7\pi}{6}} \} \quad \text{(check)}$$

$$= \{ \sqrt{3} - i, 2i, -\sqrt{3} - i \} \quad \text{(check)}$$

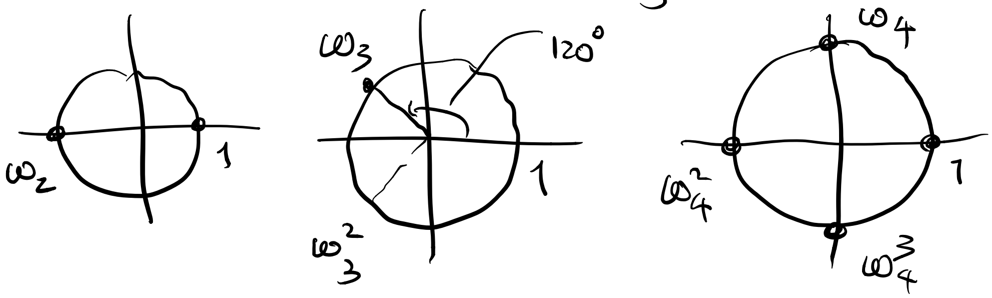


<u>eg2</u> : <u>n-roots of unity</u>

$$1^{\frac{1}{n}} = \sqrt[n]{1}\, e^{i\left(\frac{0}{n} + \frac{2k\pi}{n}\right)} = e^{i\frac{2k\pi}{n}} = \omega_n^k$$

$k = 0, 1, \cdots, n-1$

i.e. $1, \omega_n, \omega_n^2, \cdots, \omega_n^{n-1}$ are all distinct n-roots of $z = 1$



<u>eg3</u> : (Ex!) $(\sqrt{3} + i)^{1/2} = \pm \frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right)$