Review: Implicit Function Theorem and Inverse Function Theorem

Statement:

Thm (Inverse Function Theorem)

Suppose
- $f: \Omega \underset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^n$ be C^1, $f(x^0) = y^0$
- $Df(x^0)$ is invertible.

Then $\exists$ open sets V_1 containing x^0, V_2 containing y^0

and a C^1 function $g: V_2 \to V_1$ such that

(1) $g(y^0) = x^0$

(2) $g(f(x)) = x, \ \forall x \in V_1$; $f(g(y)) = y, \ \forall y \in V_2$

(inverse to each other when restricted to V_1 & V_2)

i.e. g is a local inverse of f.

(3) $Dg(y^0) = Df(x^0)^{-1}$.

Remarks: (1) $Df(x^0)$ is invertible $\Leftrightarrow$ $\det(Df(x^0)) \neq 0$.

(2) Existence of inverse is only local (in V_1), not necessary global (whole Ω)

(3) If $n=1$, "$f: \Omega \subset \mathbb{R} \to \mathbb{R}$", then

the matrix $Df(x^0) = [f'(x^0)]$ invertible

$\Leftrightarrow$ $f'(x^0) \neq 0$ ($\Rightarrow f$ strictly increasing / decreasing at x^0

& hence invertible (locally))

And the inverse $g = f^{-1}$ has derivative

$$\frac{\partial g}{\partial y}(y^0) = \frac{1}{\frac{\partial f}{\partial x}(x^0)} \quad \text{where } y_0 = f(x_0)$$

i.e.

$$Dg(y^0) = \left[\frac{\partial g}{\partial y}(y^0)\right] = \left[\frac{1}{\frac{\partial f}{\partial x}(x^0)}\right] = Df(x^0)^{-1}$$

inverse matrix

eg: $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$(x, y) \mapsto (x^2 - y^2, xy)$$

Clearly $f(-x,-y) = f(x,y)$, 2 to 1 for most points,

and hence f has no global inverse.

How about <u>local</u> inverse near a point?

Soln: Denote $f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2-y^2 \\ xy \end{bmatrix}$

Then $Df(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix}$

$\det Df(x,y) = 2x^2+2y^2 > 0 \iff (x,y) \neq (0,0)$.

Then Inverse Function Theorem $\Rightarrow$ f is locally invertible near any point $(x,y) \neq (0,0)$.

(And partial derivatives of the inverse can be found accordingly.)

Concretely, find locally inverse at $(1,2)$

let $\begin{cases} u = x^2-y^2 \\ v = xy \end{cases}$ $\left(\begin{pmatrix} x \\ y \end{pmatrix} = g\begin{pmatrix} u \\ v \end{pmatrix} = f^{-1}\begin{pmatrix} u \\ v \end{pmatrix} \right)$

Near $(1,2)$, $x \neq 0 \Rightarrow \begin{cases} y = \frac{v}{x} \text{ and} \\ u = x^2 - \left(\frac{v}{x}\right)^2 \end{cases}$

$\Rightarrow \quad x^2 = \dfrac{u \pm \sqrt{u^2+4v^2}}{2}$

Note that (i) $\begin{cases} u = u(1,2) = -3 \\ v = v(1,2) = 2 \end{cases}$

$$\Rightarrow \quad u^2+4v^2 = (-3)^2 + 4(2)^2 = 25 \neq 0$$

(ii) at $(u,v) = (-3,2)$ & $(x,y) = (1,2)$

$$1^2 = \frac{(-3) \pm \sqrt{25}}{2}$$

$\Rightarrow$ "$-$" sign should be rejected,

and $$x^2 = \frac{u+\sqrt{u^2+4v^2}}{2}$$

$$\Rightarrow \quad x = \sqrt{\frac{u+\sqrt{u^2+v^2}}{2}} \quad (\text{since } x \approx 1)$$

Then $$y = \frac{v}{\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}}$$

$\therefore$ The local inverse at $(x,y) = (1,2)$ $\left((u,v) = (-3,2)\right)$

is $$g(u,v) = \left(\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}, \frac{v}{\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}} \right)$$

if $(x,y) = (0,0)$
then $(u,v) = (0,0)$
$\Rightarrow$ no way to choose "$\pm$"!
And cannot find the function.

Statement

Thm (Implicit Function Theorem (IFT))

Let $G : \Omega \underset{\text{open}}{\subseteq} \mathbb{R}^{n+k} \to \mathbb{R}^k$ be C^1

Denote points on Ω by $(x,y) = (x_1, \cdots, x_n, y_1, \cdots, y_k)$

(i.e. $x = (x_1, \cdots, x_n)$ & $y = (y_1, \cdots, y_k)$) and represent G by

$$G(x,y) = \begin{bmatrix} g_1(x,y) \\ \vdots \\ g_k(x,y) \end{bmatrix} = \begin{bmatrix} g_1(x_1, \cdots, x_n, y_1, \cdots, y_k) \\ \vdots \\ g_k(x_1, \cdots, x_n, y_1, \cdots, y_k) \end{bmatrix}$$

Suppose $(x^0, y^0) \in \Omega$ such that
$$G(x^0, y^0) = c = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

and the $k \times k$ matrix
$$\left[\frac{\partial g_i}{\partial y_j}(x^0, y^0) \right]_{1 \le i, j \le k} = \begin{bmatrix} \frac{\partial g_1}{\partial y_1}(x^0,y^0) & \cdots & \frac{\partial g_1}{\partial y_k}(x^0,y^0) \\ \vdots & & \vdots \\ \frac{\partial g_k}{\partial y_1}(x^0,y^0) & \cdots & \frac{\partial g_k}{\partial y_k}(x^0,y^0) \end{bmatrix}$$ is invertible.

Then $\exists$ open sets $U \subseteq \mathbb{R}^n$ containing x^0 and $V \subseteq \mathbb{R}^k$ containing y^0, and a C^1 map
$$\varphi : \underset{\substack{\cup \\ x = (x_1, \cdots, x_n)}}{U^{\subset \mathbb{R}^n}} \longrightarrow \underset{\substack{\cup \\ (\varphi_1(x), \cdots, \varphi_k(x))}}{V^{\subset \mathbb{R}^k}}$$
$$x = (x_1, \cdots, x_n) \mapsto (\varphi_1(x), \cdots, \varphi_k(x)) \quad (= (y_1, \cdots, y_k))$$

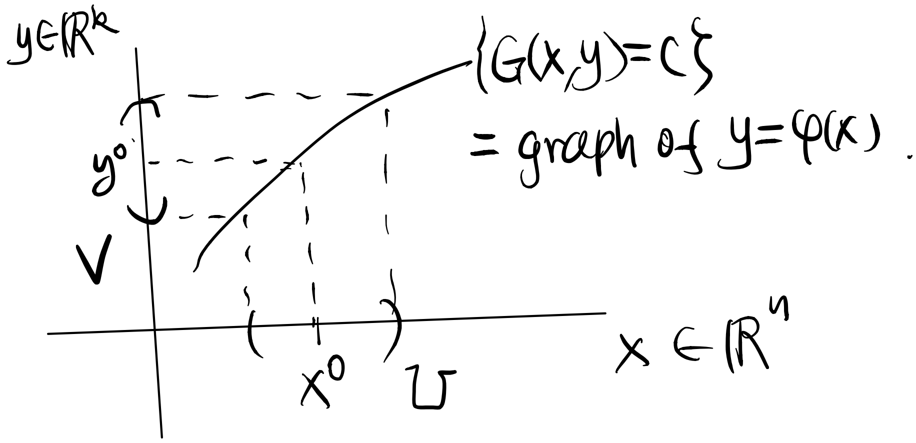
such that

(1) $\varphi(x^0) = y^0$

(2) $G(x, \varphi(x)) = c, \ \forall x \in U$.

(3)
$$\left[\frac{\partial \varphi_l}{\partial x_j}(x^0) \right]_{k \times n} = - \left[\frac{\partial g_i}{\partial y_l}(x^0, y^0) \right]^{-1}_{k \times k} \left[\frac{\partial g_i}{\partial x_j}(x^0, y^0) \right]_{k \times n}$$

(follows from implicit differentiation.)



Note: For convenient, we put all the $(y_1, \cdots, y_k)$ after $(x_1, \cdots, x_n)$ and solve $(y_1, \cdots, y_k)$ as functions of $(x_1, \cdots, x_n)$.

But, in fact, the ordering of the variables is unimportant.

Special case A : $n=2, k=1$

$$G : \Omega \subseteq \mathbb{R}^{2+1} \to \mathbb{R}$$

general notation	(x,y,z) notation
$G(x,y) = g(x_1, x_2, y) = c$	$G(x,y,z) = g(x,y,z) = c$
And $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$, $y^0 \in \mathbb{R}$ s.t. $g(x^0,y^0) = g(x_1^0, x_2^0, y^0) = c$.	And $(x_0, y_0) \in \mathbb{R}^2$, $z_0 \in \mathbb{R}$ s.t. $g(x_0, y_0, z_0) = c$
$k \times k$ matrix becomes 1×1 and is $\left[\frac{\partial g}{\partial y}(x^0,y^0)\right]$ is invertible $\Leftrightarrow$ $\frac{\partial g}{\partial y}(x^0,y^0) \neq 0$.	(1×1 matrix) $\left[\frac{\partial g}{\partial z}(x_0,y_0)\right]$ is invertible $\Leftrightarrow$ $\frac{\partial g}{\partial z}(x_0,y_0) \neq 0$
"IFT" $\frac{\partial g}{\partial y}(x^0,y^0) \neq 0$ $\Rightarrow \exists\, y = y(x_1,x_2)$ near (x_1^0, x_2^0, y^0) s.t. $\begin{cases} g(x_1,x_2,y(x_1,x_2)) = c \\ y(x_1^0,x_2^0) = y^0 \end{cases}$ (And $\frac{\partial y}{\partial x_i}(x_1^0,x_2^0)$, $i=1,2$, can be calculated using implicit differentiation.)	"IFT" $\frac{\partial g}{\partial z}(x_0,y_0) \neq 0$ $\Rightarrow \exists\, z = z(x,y)$ near (x_0,y_0,z_0) s.t. $\begin{cases} g(x,y,z(x,y)) = c \\ z(x_0,y_0) = z_0 \end{cases}$ (And $\frac{\partial z}{\partial x}(x_0,y_0)$, $\frac{\partial z}{\partial y}(x_0,y_0)$ can be calculated using implicit differentiation)

Special case B : $n=1, k=2$

$$G : \Omega \subset \mathbb{R}^{1+2} \to \mathbb{R}^2$$

General notation	(x,y,z) notation
$G(x,y_1,y_2) = \begin{bmatrix} g_1(x,y_1,y_2) \\ g_2(x,y_1,y_2) \end{bmatrix}$	$G(x,y,z) = \begin{bmatrix} g_1(x,y,z) \\ g_2(x,y,z) \end{bmatrix}$
And $x^0 \in \mathbb{R}$ & $y^0 = (y_1^0, y_2^0) \in \mathbb{R}^2$ s.t. $G(x^0,y_1^0,y_2^0) = \begin{bmatrix} g_1(x^0,y_1^0,y_2^0) \\ g_2(x^0,y_1^0,y_2^0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$	And $x_0 \in \mathbb{R}$ & $(y_0,z_0) \in \mathbb{R}^2$ s.t. $G(x_0,y_0,z_0) = \begin{bmatrix} g_1(x_0,y_0,z_0) \\ g_2(x_0,y_0,z_0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
$k \times k$ matrix is the 2×2 matrix $\begin{bmatrix} \frac{\partial g_1}{\partial y_1}(x^0,y_1^0,y_2^0) & \frac{\partial g_1}{\partial y_2}(x^0,y_1^0,y_2^0) \\ \frac{\partial g_2}{\partial y_1}(x^0,y_1^0,y_2^0) & \frac{\partial g_2}{\partial y_2}(x^0,y_1^0,y_2^0) \end{bmatrix}$	$k \times k$ matrix is the 2×2 matrix $\begin{bmatrix} \frac{\partial g_1}{\partial y}(x_0,y_0,z_0) & \frac{\partial g_1}{\partial z}(x_0,y_0,z_0) \\ \frac{\partial g_2}{\partial y}(x_0,y_0,z_0) & \frac{\partial g_2}{\partial z}(x_0,y_0,z_0) \end{bmatrix}$
"IFT" $\begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{bmatrix}^{-1}_{(x^0,y_1^0,y_2^0)}$ exists $\left(\Leftrightarrow \det \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{bmatrix}_{(x^0,y_1^0,y_2^0)} \neq 0 \right)$ $\Rightarrow$ y_1, y_2 can be written as functions of x (locally) s.t. $\begin{cases} g_1(x,y_1(x),y_2(x)) = c_1 \\ g_2(x,y_1(x),y_2(x)) = c_2 \\ y_1(x^0) = y_1^0, \ y_2(x^0) = y_2^0. \end{cases}$	"IFT" $\begin{bmatrix} \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{bmatrix}^{-1}_{(x_0,y_0,z_0)}$ exists $\left(\Leftrightarrow \det \begin{bmatrix} \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{bmatrix}_{(x_0,y_0,z_0)} \neq 0 \right)$ $\Rightarrow$ y, z can be written as functions of x (locally) s.t. $\begin{cases} g_1(x,y(x),z(x)) = c_1 \\ g_2(x,y(x),z(x)) = c_2 \\ y(x_0) = y_0, \ z(x_0) = z_0. \end{cases}$

(Partially derivatives can be calculated using implicit differentiation.)

<u>Eg</u>: Let a set $S \subseteq \mathbb{R}^3$ be defined by

$$\begin{cases} xz + \sin(zy - x^2) = 8 \\ x + 4y + 3z = 18 \end{cases}$$

Near the point $(2,1,4)$, can we solve 2 of variables as functions of the remaining variable ?

<u>Soln</u>: Let $\begin{cases} g_1(x,y,z) = xz + \sin(zy - x^2) \\ g_2(x,y,z) = x + 4y + 3z \end{cases}$

Calculate the matrix, at $(2,1,4)$,

$$\begin{bmatrix} - \nabla g_1 - \\ - \nabla g_2 - \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} z - 2x\cos(zy - x^2) & z\cos(zy - x^2) & x + y\cos(zy - x^2) \\ 1 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix} \quad \text{at } (2,1,4)$$

Check 3 cases: (x,y) (y,z)

(x,z)

(i) The 2X2 matrix corresponding to (x,y) variables has

$$\begin{vmatrix} 0 & 4 \\ 1 & 4 \end{vmatrix} = -4 \neq 0$$

IFT $\Rightarrow$ x, y can be written as functions of z

such that $\begin{cases} g_1(x(z), y(z), z) = c_1 \\ g_2(x(z), y(z), z) = c_2 \\ x(4) = 2, \; y(4) = 1 \end{cases}$ locally

(ii) The 2X2 matrix corresponding to (x,z) variables has

$$\begin{vmatrix} 0 & 3 \\ 1 & 3 \end{vmatrix} = -3 \neq 0$$

IFT $\Rightarrow$ x, z can be written as functions of y

such that

$$\begin{cases} g_1(x(y), y, z(y)) = c_1 \\ g_2(x(y), y, z(y)) = c_2 \\ x(1) = 2, \; z(1) = 4 \end{cases}$$

(iii) The 2X2 matrix corresponding to (y,z) variables has

$$\begin{vmatrix} 4 & 3 \\ 4 & 3 \end{vmatrix} = 0$$

IFT <u>doesn't</u> apply & we have no conclusion.

If fact : In this particular example,

if $y=y(x)$ & $z=z(x)$ near $(2,1,4)$.

Then implicit differentiation $\Rightarrow$

$$\begin{cases} 0+4\dfrac{dy}{dx}+3\dfrac{dz}{dx}=0 \\ 1+4\dfrac{dy}{dx}+3\dfrac{dz}{dx}=0 \end{cases} \quad \text{at } x=2 \quad (y=1, z=4)$$

which is a contradiction.

$\Rightarrow$ We cannot find "differentiable" functions $y=y(x)$ & $z=z(x)$ solving the constraints.