Proof of Divergence Thm

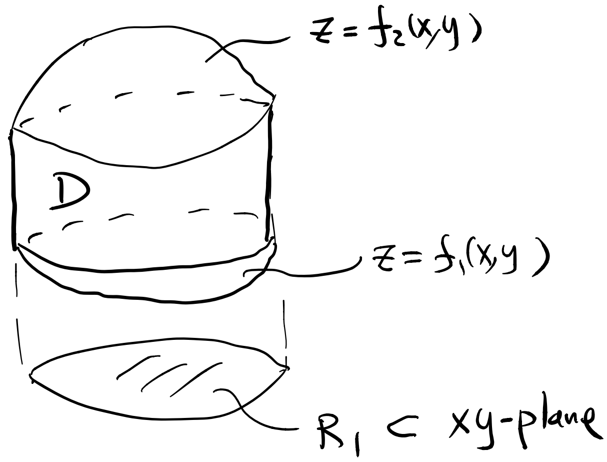
Same as Green's Thm, we'll prove only the case of special domain D which is of type I, II, and III:

$$D = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R_1,\ f_1(x,y) \le z \le f_2(x,y)\} \quad (\text{type I})$$

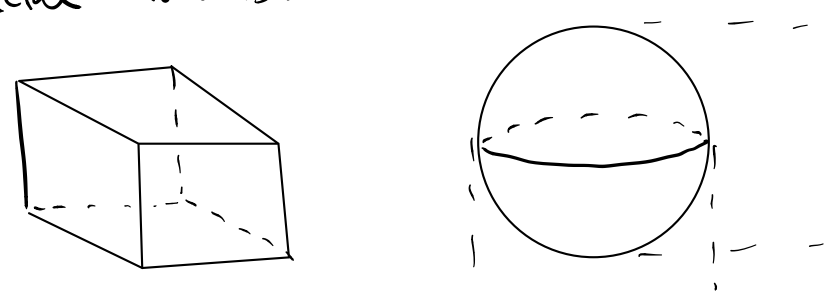
$$= \{(x,y,z) \in \mathbb{R}^3 : (y,z) \in R_2,\ g_1(y,z) \le x \le g_2(y,z)\} \quad (\text{type II})$$

$$= \{(x,y,z) \in \mathbb{R}^3 : (x,z) \in R_3,\ h_1(x,z) \le y \le h_2(x,z)\} \quad (\text{type III})$$

eg: Type I domain



eg: Special domains:



And also as in the proof of Green's Thm,

for $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

we'll prove 3 equalities in the following which combine to give the divergence thm :

$$\begin{cases} \displaystyle\iint_S M\hat{i}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial M}{\partial x}\,dV & \text{(by type II)} \\[2ex] \displaystyle\iint_S N\hat{j}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial N}{\partial y}\,dV & \text{(by type III)} \\[2ex] \displaystyle\iint_S L\hat{k}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial L}{\partial z}\,dV & \text{(by type I)} \end{cases}$$

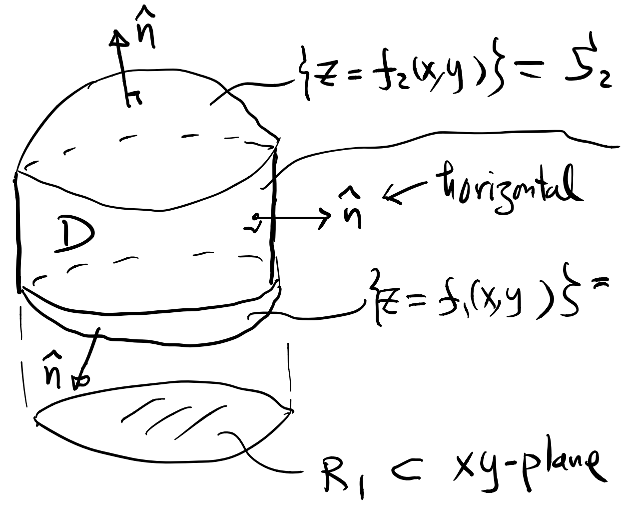
The proofs are similar, we'll prove only the last one:

$$\iint_S L\hat{k}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial L}{\partial z}\,dV$$

By Fubini's Thm

$$\text{R.H.S.} = \iiint_D \frac{\partial L}{\partial z}\,dV = \iint_{R_1}\left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial L}{\partial z}\,dz\right]dx\,dy \quad \text{(by type I)}$$

$$= \iint_{R_1}\left[L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))\right]\underline{dx\,dy}.$$



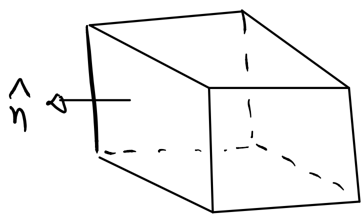
For the L.H.S., we note that by definition of type I domain, the boundary surface S of D can be written as

$$S = S_1 \cup S_2 \cup S_3,$$

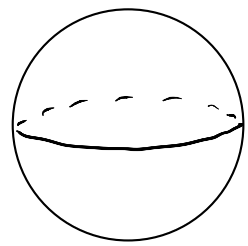
where $S_1 = \text{graph of } f_1 = \{(x, y, f_1(x,y))\} = \{z = f_1(x,y)\}$

$S_2 = \text{graph of } f_2 = \{(x, y, f_2(x,y))\} = \{z = f_2(x,y)\}$

$S_3 =$ a vertical surface (which could be empty) between S_1 & S_2



$S_3 =$ union of 4 vertical squares



$S_3 =$ empty

Hence

$$\text{L.H.S.} = \iint_S L\hat{k}\cdot\hat{n}\,d\sigma = \iint_{S_1} L\hat{k}\cdot\hat{n}\,d\sigma + \iint_{S_2} L\hat{k}\cdot\hat{n}\,d\sigma$$

$$+ \iint_{S_3} L\cancel{\hat{k}\cdot\hat{n}}\,d\sigma$$

(since $\hat{n}$ of a vertical surface is horizontal, hence $\hat{k}\cdot\hat{n} = 0$)

Now on the upper surface $S_2 = \{z = f_2(x,y)\}$,

the outward normal $\hat{n}$ is upward (in the sense that $\hat{n}\cdot\hat{k} > 0$)

Note that the parametrization

$$(x,y) \mapsto \vec{r}(x,y) = x\hat{i} + y\hat{j} + f_2(x,y)\hat{k}$$

has
$$\begin{cases} \vec{r}_x = \hat{i} + \dfrac{\partial f_2}{\partial x}\hat{k} \\[2mm] \vec{r}_y = \hat{j} + \dfrac{\partial f_2}{\partial y}\hat{k} \end{cases}$$

and
$$\vec{r}_x \times \vec{r}_y = -\frac{\partial f_2}{\partial x}\hat{i} - \frac{\partial f_2}{\partial y}\hat{j} + \hat{k}$$

+ve $\Rightarrow$ $\vec{r}_x \times \vec{r}_y$ is upward.

Hence
$$\hat{n} = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$$
is the upward normal

and
$$\hat{k}\cdot\hat{n} = \frac{1}{|\vec{r}_x \times \vec{r}_y|}.$$

Therefore
$$\iint_{S_2} L\,\hat{k}\cdot\hat{n}\,d\sigma = \iint_{R_1} L(x,y,f_2(x,y)) \cdot \overbrace{\frac{1}{\cancel{|\vec{r}_x \times \vec{r}_y|}}}^{\hat{k}\cdot\hat{n}} \cdot \overbrace{\cancel{|\vec{r}_x \times \vec{r}_y|}\,dA}^{d\sigma}$$

$$= \iint_{R_1} L(x,y,f_2(x,y))\,dx\,dy$$

Similarly, note that the <u>outward</u> normal on S_1 (lower surface) is <u>downward</u> (i.e. $\hat{n}\cdot\hat{k} < 0$), we have (by similar calculation)

$$\hat{n} = -\frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}, \quad \text{where } \vec{r}(x,y) = x\hat{i} + y\hat{j} + f_1(x,y)\hat{k}$$

$$\Rightarrow \quad \hat{k}\cdot\hat{n} = -\frac{1}{|\vec{r}_x \times \vec{r}_y|} \qquad \text{(check!)}$$

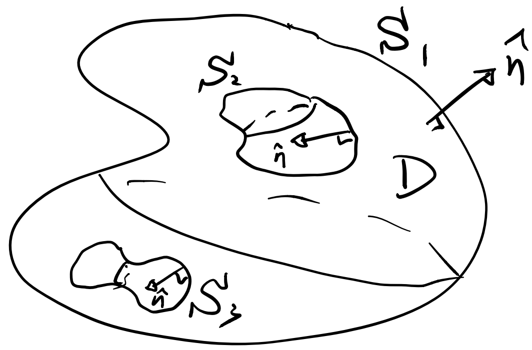
Hence $\displaystyle\iint_{S_1} L\hat{k}\cdot\hat{n}\,d\sigma = -\iint_{R_1} L(x,y,f_1(x,y))\,dxdy$

$$\therefore \iint_{S} L\hat{k}\cdot\hat{n}\,d\sigma = \iint_{R_1}\left[L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))\right]dxdy$$

$$= \iiint_D \frac{\partial L}{\partial z}\,dV .$$

This completes the proof of the divergence thm. #

Note: Similar to Green's Thm, the Divergence Thm is also hold for solid region with finitely many holes insides:

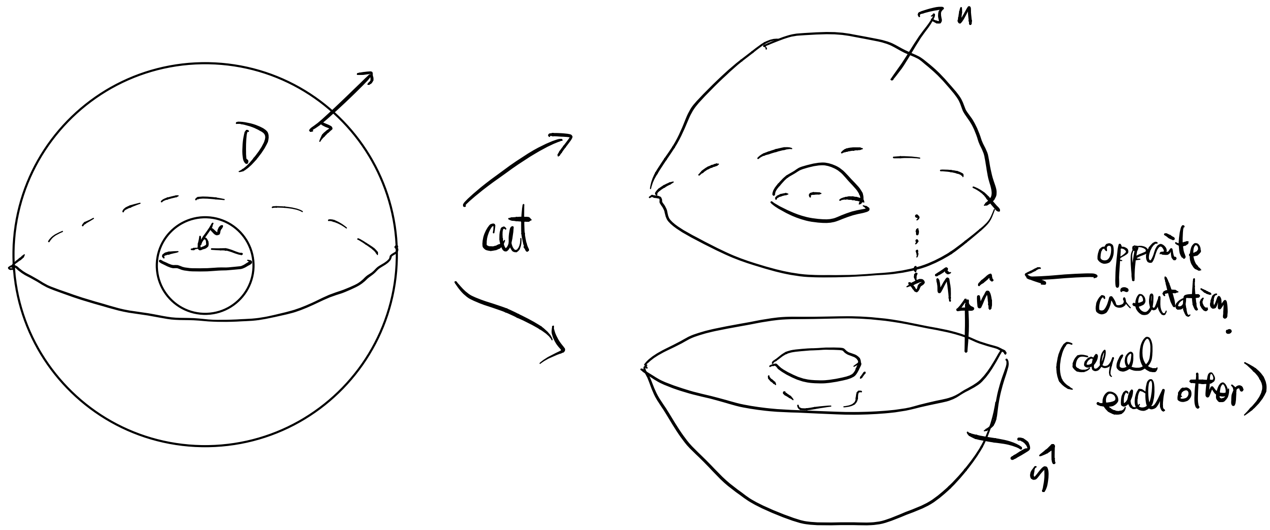


D = solid region inside S_1 but outside of S_2 and S_3

$$\boxed{\iiint_D \vec{\nabla}\cdot\vec{F}\,dV = \sum_{i=1}^{n}\iint_{S_i} \vec{F}\cdot\hat{n}\,d\sigma}$$

for $\hat{n}$ = <u>outward</u> normal <u>with respect to D</u>.

eg (Idea of proof of this kind of surface :)



(see ex. 14 of §16.8 in Thomas' Calculus for an explicit example. (HW10))

<u>Note</u>: Physical meaning of $\operatorname{div} F = \vec{\nabla} \cdot \vec{F}$ in $\mathbb{R}^3$

= <u>flux density</u> (by the divergence thm)

The coverage of the take-home final is up to "here":

divergence thm.

(Including all materials in the Lecture notes, tutorial notes, HW assignments, text book & reference books, with emphasis on <u>vector analysis part</u>) (30 hours)
