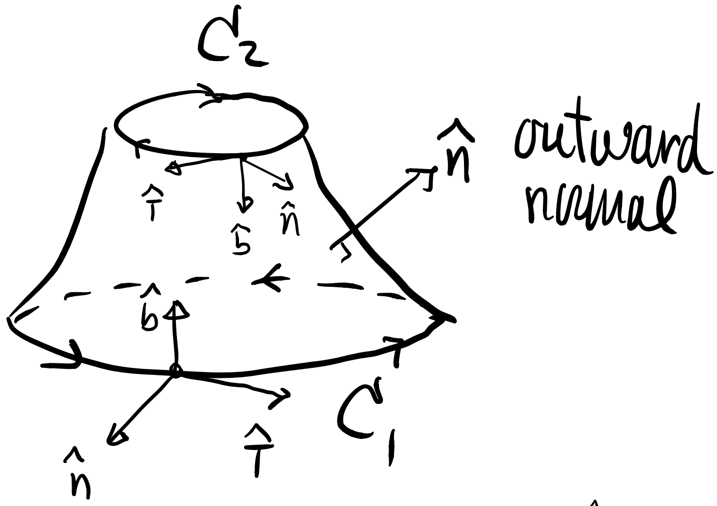
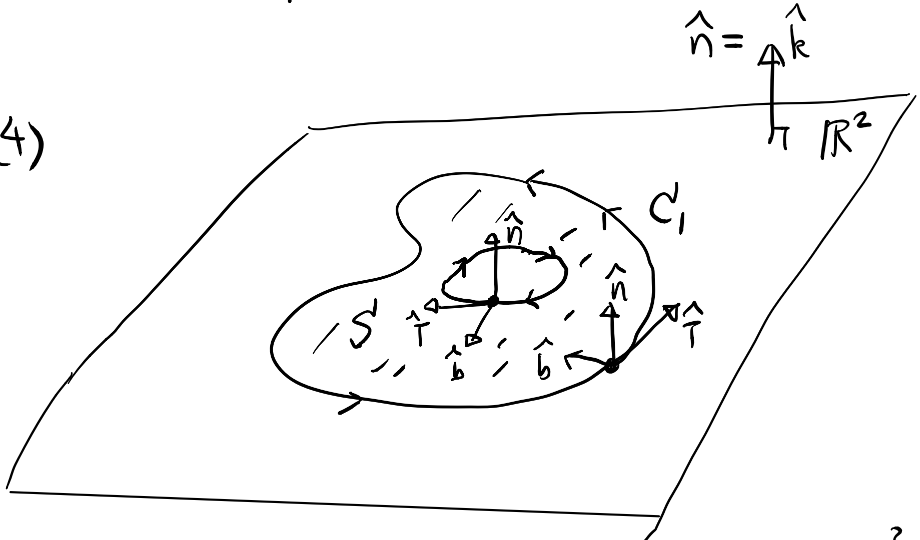
eg60 (cont'd)

(3)



(4)



$(S' \subset \mathbb{R}^2)$
$(\hat{n} = \hat{k})$

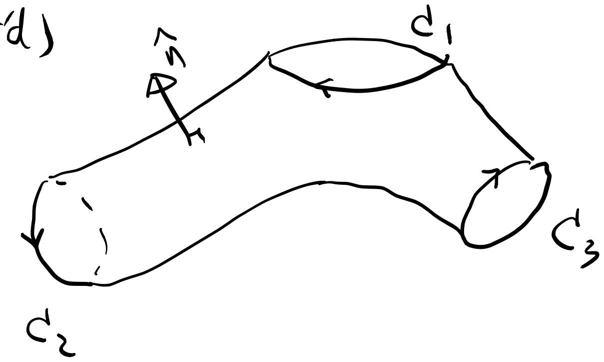
Important remark: If S' is a region in $\mathbb{R}^2$, then a boundary component of S' (C_1' or C_2' for instance) has "2" concepts of "oriented anti-clockwisely" with respect to $\begin{cases} S' = \text{region} \\ \mathbb{R}^2 \end{cases}$

Even S' and $\mathbb{R}^2$ have the same orientation, i.e. $\hat{n} = \hat{k}$, we still have the following situations: (C_1', C_2' as in figure)

	S' (region)	$\mathbb{R}^2$
C_1'	anti-clockwise (+)	anti-clockwise (+)
C_2'	anti-clockwise (+)	clockwise (−)

eg 60 (cont'd)

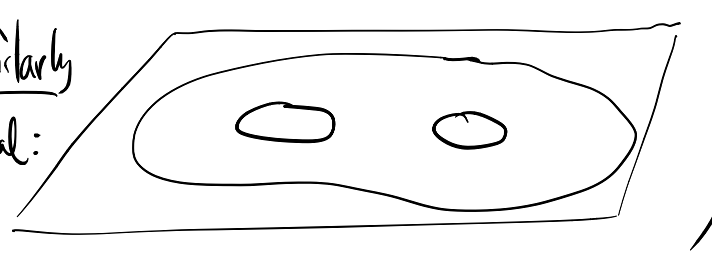
(5)



what is the oriented of C_i s.t. their oriented anti-clockwise with respect to $\hat{n}$?

(Ex!)

(Similarly optional:

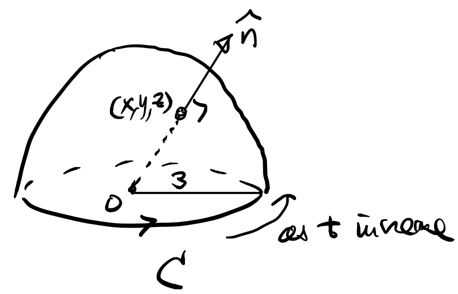

)

eg 61 Verifying Stokes' Thm

(a) $S_1 : x^2+y^2+z^2=9, \ z \geq 0$

with upward normal $\hat{n}$

boundary $C : x^2+y^2=9, \ z=0$



$C: \vec{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle \ , \ 0 \leq t \leq 2\pi$

$= 3\cos t \,\hat{i} + 3\sin t \,\hat{j}$

(has the correct direction, i.e. oriented anti-clockwisely wrt $\hat{n}$)

Suppose $\vec{F} = y\hat{i} - x\hat{j}$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (3\sin t\,\hat{i} - 3\cos t\,\hat{j}) \cdot (-3\sin t\,\hat{i} + 3\cos t\,\hat{j})\,dt$$

(check!)

$= -18\pi$

For the surface integral:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\hat{k} \quad \text{(check!)}$$

Since S_1 is a hemisphere (upper) centered at origin

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{3}(x\hat{i}+y\hat{j}+z\hat{k})$$

($z \geq 0$ $\Leftrightarrow$ upward)

The surface S_1 can be regarded as a level surface given by

$$g(x,y,z) = x^2+y^2+z^2 = 9$$

Note: $\vec{\nabla}g = (2x, 2y, 2z)$

Since $z>0$ (except the boundary) on S_1,

$$\frac{\partial g}{\partial z} = 2z \neq 0$$

Hence $$d\sigma = \frac{|\vec{\nabla}g|}{\left|\frac{\partial g}{\partial z}\right|}\,dxdy = \frac{\sqrt{4x^2+4y^2+4z^2}}{|2z|}\,dxdy = \frac{3}{z}\,dxdy$$

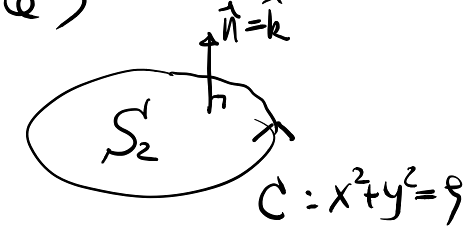
(since $z>0$)

Therefore $$\iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$$

$$= \iint_{\{x^2+y^2\leq 9\}} (-2\hat{k})\cdot\frac{1}{3}(x\hat{i}+y\hat{j}+z\hat{k})\cdot\frac{3}{z}\,dxdy$$

$$= \iint_{\{x^2+y^2\leq 9\}} (-2)\,dxdy = -18\pi \quad \text{(check!)}$$

(b) (Same C & same $\vec{F}$, but new surface)

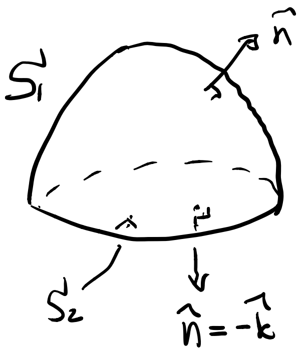


$S_2 : x^2+y^2 \le 9, \ z=0$

$$\iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = \iint_{\{x^2+y^2\le 9\}} (-2\hat{k})\cdot\hat{k}\, dxdy$$

$$= -2 \text{ Area}(\{x^2+y^2\le 9\}) = -18\pi \quad \text{(check!)}$$

(c) Same $\vec{F} = y\hat{i} - x\hat{j}$



$S_3 = S_1 \cup S_2$

S_3 has no boundary and in fact encloses a solid region.

Suppose $\hat{n}$ = outward normal of the solid.

$$\iint_{S_3} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = \iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma + \iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma$$

$$= \iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma + \iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot(-\hat{k})\, d\sigma$$

$$= -18\pi - \iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{k}\, d\sigma$$

$$= -18\pi - (-18\pi)$$

$$= 0$$

$\left(S_3 \text{ has no boundary} \Rightarrow \oint_{\partial S_3} \vec{F}\cdot d\vec{r} = 0 \right)$