By $\vec{\nabla} F \neq 0$, at least one of the partial derivatives F_x, F_y, & F_z is nonzero. Let assume $F_z = \frac{\partial F}{\partial z} \neq 0$ (the other cases are similar)

IFT $\Rightarrow$ $S = F^{-1}(c) = \{F(x,y,z) = c\}$

can be written (locally) as a graph

$$z = f(x,y) \qquad \text{(near a point)}$$

i.e. $$F(x,y,f(x,y)) = c \qquad \text{(near a point)}$$

Then chain rule $\Rightarrow$ $$\begin{cases} f_x = -\dfrac{F_x}{F_z} \\ f_y = -\dfrac{F_y}{F_z} \end{cases} \qquad (F_z \neq 0)$$

Hence $\text{Area}(S) = \iint_{\Omega} \sqrt{1+f_x^2+f_y^2}\, dA$ where Ω = domain of the (local) $z=f(x,y)$.

$$= \iint_{\Omega} \sqrt{1+\frac{F_x^2}{F_z^2}+\frac{F_y^2}{F_z^2}}\, dxdy$$

(↑ practically may be hard to find)

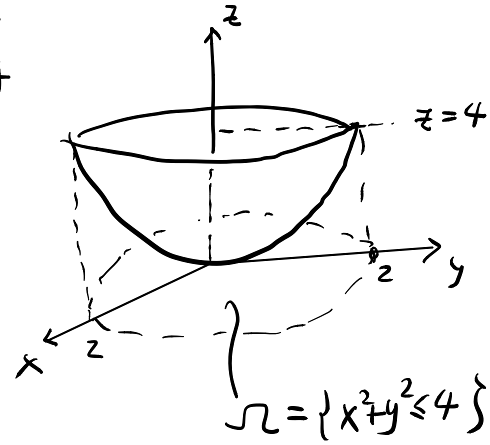
$$= \iint_{\Omega} \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|}\, dxdy$$

Thm 12 If $S = F^{-1}(c)$ is a smooth level surface such that $F_z \neq 0$, and can be represented by an implicit function over a domain Ω.

Then
$$\text{Area}(S) = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dA = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dxdy$$

(Similar for the cases that $F_x \neq 0$ or $F_y \neq 0$)

eg 54: Find surface area of the paraboloid $x^2+y^2-z=0$ below $z=4$



(This is in fact a graph, but we do it using method of level surface)

Soln: Let $F(x,y,z) = x^2+y^2-z$

For $z=4$, $x^2+y^2-z=0 \Rightarrow x^2+y^2=4$

$\Rightarrow$ projected region $\Omega = \{(x,y): x^2+y^2 \leq 4\}$

Check: $\vec{\nabla} F = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$\Rightarrow \quad F_z = -1 \neq 0, \quad \forall (x,y) \in \Omega$

$$\therefore \text{Surface Area} = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dA$$

$$= \iint_{\Omega} \frac{\sqrt{(2x)^2+(2y)^2+1}}{|-1|}\, dxdy$$

$$= \iint_{\{x^2+y^2\leq 4\}} \sqrt{4(x^2+y^2)+1}\, dxdy$$

(using polar coordinates)

(check!)

$$= \frac{\pi}{6}\left[(\sqrt{17})^3 - 1\right]$$

#

Def 16 Surface Integral (of a function)

Suppose $G : \tilde{S} \to \mathbb{R}$ is a continuous function on a surface $\tilde{S}$, parametrized by $\vec{F}(u,v)$, $(u,v) \in R$ (region R). Then the integral of G on $\tilde{S}$ is

$$\iint_{\tilde{S}} G \, d\sigma \stackrel{\text{def}}{=\!=} \iint_{R} G(\vec{F}(u,v)) \, |\vec{F}_u \times \vec{F}_v| \, dA$$

area element of $\tilde{S}$

element area of the parameter space $dA = du\,dv$

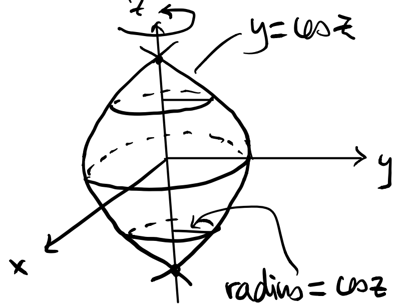
Note: In the cases of graph or level surface, we have

(i) $$\iint_{\tilde{S}} G \, d\sigma = \iint_{(x,y)} G(x,y,f(x,y)) \sqrt{1 + |\vec{\nabla} f|^2} \, dx\,dy$$
(for $z = f(x,y)$)

(ii) $$\iint_{\tilde{S}} G \, d\sigma = \iint_{(x,y)} G(x,y,z) \frac{|\vec{\nabla} F|}{|F_z|} \, dx\,dy$$
(for $F(x,y,z) = c$, $F_z \neq 0$)

(may be difficult to find these: region & z in terms of (x,y))

eg 56 (a surface of revolution of the curve $y = \cos z$)



$\left(-\frac{\pi}{2} \le z \le \frac{\pi}{2}\right)$

Let $G(x,y,z) = \sqrt{1 - x^2 - y^2}$ be a function on $\tilde{S}$

Find $\iint_{\tilde{S}} G \, d\sigma$.

Soln: S can be parametrized by

$$\begin{cases} x = \cos z \cos\theta, \\ y = \cos z \sin\theta, \\ z = z \end{cases} \qquad \begin{array}{l} -\pi \le \theta \le \pi \\ -\frac{\pi}{2} \le z \le \frac{\pi}{2} \end{array}$$

i.e. $\vec{r}(\theta, z) = \cos z \cos\theta\,\hat{i} + \cos z \sin\theta\,\hat{j} + z\,\hat{k}$

(Note: there is an exceptional set of "1-dim" which is not a parametric surface corresponds to $\theta = \pi$ or $-\pi$, $z = -\frac{\pi}{2}$ or $\frac{\pi}{2}$)

$$\Rightarrow \begin{cases} \vec{r}_\theta = -\cos z \sin\theta\,\hat{i} + \cos z \cos\theta\,\hat{j} \\ \vec{r}_z = -\sin z \cos\theta\,\hat{i} - \sin z \sin\theta\,\hat{j} + \hat{k} \end{cases}$$

$\Rightarrow \quad \vec{r}_\theta \times \vec{r}_z = \cos z \cos\theta\,\hat{i} + \cos z \sin\theta\,\hat{j} + \sin z \cos z\,\hat{k}$ (check!)

$\Rightarrow \quad |\vec{r}_\theta \times \vec{r}_z| = \sqrt{\cos^2 z\,(1 + \sin^2 z)} = \cos z \sqrt{1 + \sin^2 z}$

($\cos z \ge 0$ for $-\frac{\pi}{2} \le z \le \frac{\pi}{2}$)

Then

$$\iint_S G\,d\sigma = \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} G(\vec{r}(\theta, z))\,|\vec{r}_\theta \times \vec{r}_z|\,dz\,d\theta$$

$$= \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - x^2 - y^2}\,\cos z \sqrt{1 + \sin^2 z}\,dz\,d\theta$$

$$= \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \cos^2 z} \cdot \cos z \sqrt{1 + \sin^2 z}\,dz\,d\theta$$

check

$$= \cdots = 2 \int_{-\pi}^{\pi} \int_{0}^{\frac{\pi}{2}} \sin z \cos z \sqrt{1 + \sin^2 z}\,dz\,d\theta = \frac{4\pi}{3}(2\sqrt{2} - 1)$$

Check!

#

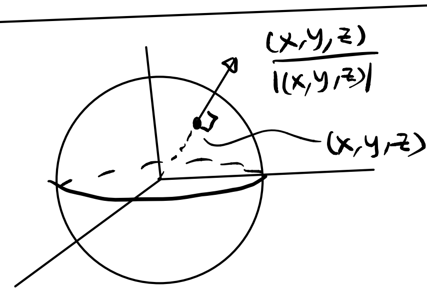
Orientation of Surfaces

To integrate vector fields over surfaces, we need

Def 17 (Orientation of a surface in $\mathbb{R}^3$)

A surface S is orientable if one can define a unit normal vector field continuously at every point of S.

eg 57: (i) $S^2 = \{x^2+y^2+z^2=1\}$

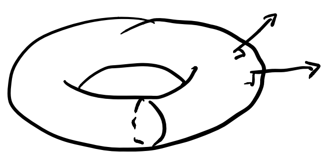


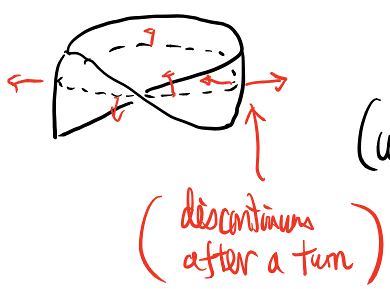
$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= x\hat{i}+y\hat{j}+z\hat{k} \text{ on } S^2$$

is a continuous unit normal vector field on S^2

$\Rightarrow$ S^2 is orientable.

(ii)  Torus is orientable

(iii) 

Möbius strip is not orientable (usually refered as surface of one side)

Remark: Parametric surface $S: \vec{r}(u,v)$ are always orientable:

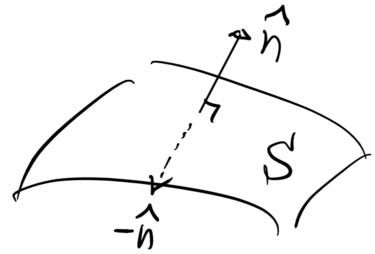
$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad \text{is a continuous unit normal vector field on } S.$$

($\vec{r}_u, \vec{r}_v$ "continuous" tangent vectors $\Rightarrow$ $\vec{r}_u \times \vec{r}_v$ is "continuous" normal vector

$|\vec{r}_u \times \vec{r}_v| \neq 0 \quad \Rightarrow \quad \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ continuous unit normal vector field)

Terminology

Given a connected orientable surface $S \subset \mathbb{R}^3$, there are two ways to assign the continuous unit normal vector field



Suppose S is orientable and we have chosen one continuous unit normal vector field $\hat{n}$.

(independent of the parametrization at the beginning)

Ref 18: We said that a parametrization $\vec{r}(u,v)$ of S is compatible with the orientation of S given by the unit normal vector field $\hat{n}$,

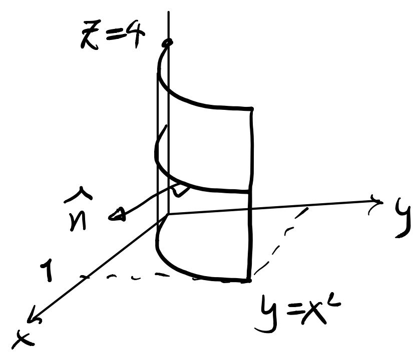
$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

(one usually refer the chosen unit normal vector field as the orientation of S,)

Def 19: Let S be orientable with unit normal $\hat{n}$ (continuous). Let $\vec{F}$ be a vector field on S. Then the flux of $\vec{F}$ across S is

$$\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\, d\sigma$$

eg 59: $S: y = x^2$ $\quad 0 \le x \le 1$, $0 \le z \le 4$



with $\hat{n}$ given by the natural parametrization

$$\vec{r}(x,z) = x\hat{i} + x^2\hat{j} + z\hat{k}$$

$$\begin{cases} \vec{r}_x = \hat{i} + 2x\hat{j} \\ \vec{r}_z = \hat{k} \end{cases} \Rightarrow \vec{r}_x \times \vec{r}_z = (\hat{i} + 2x\hat{j}) \times \hat{k} = 2x\hat{i} - \hat{j}$$

$$\Rightarrow \quad \hat{n} = \frac{2x\hat{i} - \hat{j}}{\sqrt{4x^2+1}}$$

Let $\vec{F} = yz\hat{i} + x\hat{j} - z^2\hat{k}$

Find $\iint_S \vec{F}\cdot\hat{n}\, d\sigma$

Soln: $$\iint_S \vec F\cdot\hat n\,d\sigma = \int_0^4\int_0^1 \underbrace{(yz\hat i + x\hat j - z^2\hat k)}_{\vec F}\cdot \underbrace{\frac{2x\hat i-\hat j}{\cancel{\sqrt{4x^2+1}}}}_{\hat n}\cdot \underbrace{\cancel{\sqrt{4x^2+1}}\,dx\,dz}_{d\sigma}$$

$$= \int_0^4\int_0^1 (2x^3z - x)\,dx\,dz \quad \text{(check!)}$$

$$= 2 \quad \text{(check!)}$$

Remark: $$\iint_S \vec F\cdot\hat n\,d\sigma = \iint_{(u,v)} \vec F(\vec r(u,v))\cdot \frac{\vec r_u\times\vec r_v}{|\vec r_u\times\vec r_v|}\cdot|\vec r_u\times\vec r_v|\,du\,dv$$

$$= \iint_{(u,v)} \vec F(\vec r(u,v))\cdot(\vec r_u\times\vec r_v)\,du\,dv.$$

Thm 12 (Stokes' Theorem)

Let S be a piecewise smooth oriented surface with piecewise smooth boundary C (including the case that C is a union of finitely many curves). Let

$$\vec F = M\hat i + N\hat j + L\hat k \quad \text{be a } C^1 \text{ vector field.}$$

Suppose C is oriented anti-clockwisely with respect to the unit normal vector field $\hat n$ on S. Then

$$\oint_C \vec F\cdot d\vec r = \iint_S \operatorname{curl}\vec F\cdot\hat n\,d\sigma$$

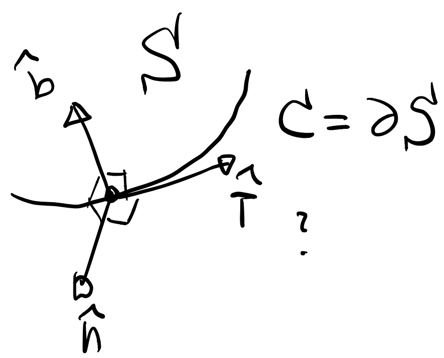
$$= \iint_S (\vec\nabla\times\vec F)\cdot\hat n\,d\sigma$$

Here: (i) if $C = C_1 \cup \cdots \cup C_k$, then it means

$$\sum_{i=1}^{k} \oint_{C_i} \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma$$

(ii) "C is oriented anti-clockwisely with respect to the unit normal vector field $\hat{n}$" means that

we choose the direction of C such that its (unit) tangent vector $\hat{T}$ satisfies



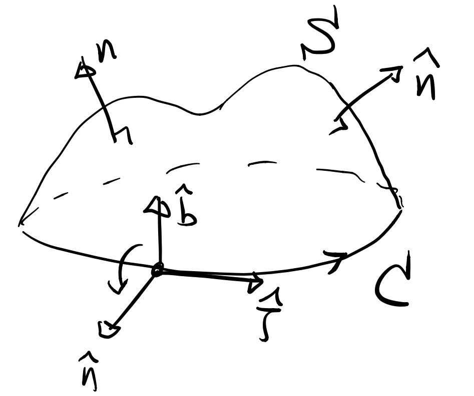
$$\boxed{\hat{b} = \hat{n} \times \hat{T} \text{ pointing toward the surface } S}$$

i.e. $\hat{b}$ is a (unit) tangent vector to S and normal to C and pointing toward S. Then

$$\boxed{\hat{T} = \hat{b} \times \hat{n}}$$

eg 60

(1)



(2) If $S \subset \mathbb{R}^2$ with $\hat{n} = \hat{k}$

same as the usual

anti-clockwise direction

of a closed plane curve!