Def 12: The divergence of $\vec{F} = M\hat{i} + N\hat{j}$ is defined to be

$$\operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Note: $$\operatorname{div}\vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \iint_{\bar{D}_\varepsilon(x,y)} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \oint_{\partial \bar{D}_\varepsilon(x,y)} \vec{F} \cdot \hat{n}\, ds$$

called "flux density".

Notation: For $f(x,y)$, $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ (gradient)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) f$$

It is convenient to denote

$$\boxed{\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}}$$

Then $$\vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) \cdot \left(M\hat{i} + N\hat{j}\right)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \operatorname{div}\vec{F}$$

Hence we also write $\boxed{\operatorname{div}\vec{F} = \vec{\nabla} \cdot \vec{F}}$

Def 13: Define $\operatorname{rot}\vec{F}$ to be

$$\operatorname{rot}\vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \qquad (\text{for } \vec{F} = M\hat{i} + N\hat{j})$$

Note: $\operatorname{rot}\vec{F} = \lim_{\varepsilon\to 0} \dfrac{1}{\operatorname{Area}(\bar{D}_\varepsilon(x,y))} \iint_{\bar{D}_\varepsilon(x,y)} \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) dA$

$$= \lim_{\varepsilon\to 0} \frac{1}{\operatorname{Area}(\bar{D}_\varepsilon(x,y))} \oint_{\partial\bar{D}_\varepsilon(x,y)} \vec{F}\cdot\hat{T}\, ds$$

(called)

$= $ circulation density

Using $\vec{\nabla} = \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y}$, we can write

$$\boxed{\operatorname{rot}\vec{F} = (\vec{\nabla}\times\vec{F})\cdot\hat{k}}$$

Since $\vec{F} = M\hat{i} + N\hat{j} + 0\cdot\hat{k}$ (in $\mathbb{R}^3$) $\left(M = M(x,y) \text{ \& } N = N(x,y)\right)$

$\vec{\nabla} = \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}$ (in $\mathbb{R}^3$) $\left(\dfrac{\partial M}{\partial z} = \dfrac{\partial N}{\partial z} = 0\right)$

$$\Rightarrow \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ M & N \end{vmatrix}\hat{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

(check!)

$\Rightarrow \operatorname{rot}\vec{F} = (\vec{\nabla}\times\vec{F})\cdot\hat{k}$ i.e. $\hat{k}$-component of $\underline{\vec{\nabla}\times\vec{F}}$.

A name for $\vec{\nabla}\times\vec{F}$ is $\operatorname{curl}\vec{F}$: $\boxed{\operatorname{curl}\vec{F} \overset{\text{def}}{=} \vec{\nabla}\times\vec{F}}$

In these notation, the Green's thm can be written as

Vector form of Green's Thm

normal form

$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA}$$

or

$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{\nabla}\cdot\vec{F}\,dA}$$

tangential form

$$\boxed{\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_R \operatorname{curl}\vec{F}\cdot\hat{k}\,dA}$$

or

$$\boxed{\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_R (\vec{\nabla}\times\vec{F})\cdot\hat{k}\,dA}$$

And Thm 10 can be written as

Thm 10': Ω simply-connected & connected, $\vec{F}\in C^1$.

Then $\vec{F}$ = conservative $\Leftrightarrow$ $\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = 0$

(check: case for $n=3$)

Note: (i) $\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F}$ defined only in $\mathbb{R}^3$ ($\supset \mathbb{R}^2$)

(ii) but $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$ can be defined on $\mathbb{R}^n$ for any n

In particular, in $\mathbb{R}^3$

Def 12' The divergence of $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ is defined to be

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(M\hat{i}+N\hat{j}+L\hat{k}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial L}{\partial z}$$

Then one can easily check the following facts: (Ex!)

(i) $\vec{\nabla}\times(\vec{\nabla} f)=0$ (i.e. curl $\vec{\nabla} f=0$)

(ii) $\vec{F}$ conservative $\Rightarrow$ curl $\vec{F}=\vec{\nabla}\times\vec{F}=0$

(iii) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})=0$ (i.e. div(curl $\vec{F}$)$=0$)

<u>Remark</u>: $\vec{\nabla}\cdot(\vec{\nabla} f)\neq 0$ in general, and it is called the <u>Laplacian</u> of f and is denoted by

$$\boxed{\begin{aligned}\vec{\nabla}^2 f &= \vec{\nabla}\cdot(\vec{\nabla} f)=\operatorname{div}(\vec{\nabla} f)\\ &= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\end{aligned}}$$

[In graduate level, it will be denoted by $\Delta=\vec{\nabla}^2$ or $\Delta=-\vec{\nabla}^2$]

The "operator" $\vec{\nabla}^2$ is called the <u>Laplace operator</u> and the equation $\vec{\nabla}^2 f=0$ is called the <u>Laplace equation</u>. Solutions to the Laplace equation are called <u>harmonic functions</u>.

<u>Pf of Thm 10 ($n=2$)</u>

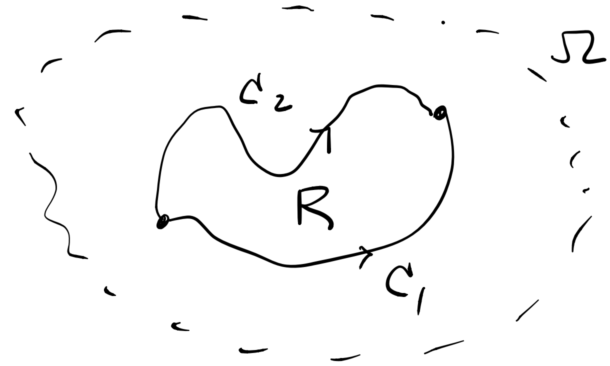
We only need to that if Ω simply-connected (& connected) and

$$\vec{\nabla}\times\vec{F}=0 \quad \left(\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\right)$$

then $\vec{F}$ is conservative.

<u>Case 1</u>: C_1, C_2 have no intersection. (except the same end points)

Then "Ω is simply-connected"

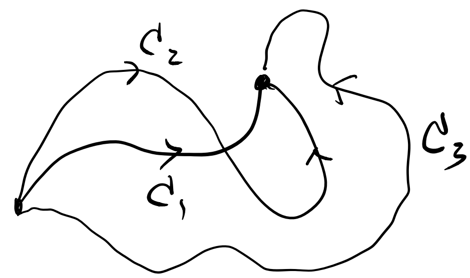


$\Rightarrow$ the region R enclosed by C_1 and C_2 lies completely inside Ω.

Then by Green's Thm

$$0 = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \pm \left(\int_{C_1} - \int_{C_2}\right)(M\,dx + N\,dy)$$

$$\Rightarrow \quad \int_{C_1} M\,dx + N\,dy = \int_{C_2} M\,dx + N\,dy$$



Case 2 C_1, C_2 intersect

Pick another curve C_3 with the same starting point and end point, and does not intersect C_1 or C_2.

Then by case 1, $\int_{C_1} M\,dx + N\,dy = \int_{C_3} M\,dx + N\,dy$

$$= \int_{C_2} M\,dx + N\,dy$$

$\therefore$ $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path and hence $\vec{F}$ is conservative. #

In order to apply Green's Thm to more general situations, we need a <u>general form of Green's Thm</u>:

Suppose that we have a simple closed curve C in $\mathbb{R}^2$

Suppose that $C_1, C_2, \cdots, C_n$ be pairwise disjoint, piecewise smooth, simple closed curves, such that $C_1, \cdots, C_n$ are enclosed by C.

(All $C, C_1, \cdots, C_n$ are anti-clockwise oriented.)

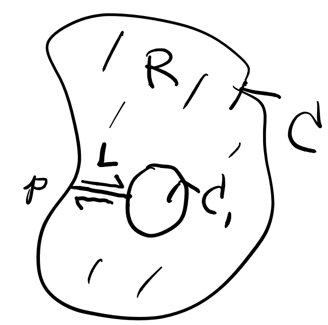
Let R be the region between C and $C_1, \cdots, C_n$.

Suppose that $\vec{F} = M\hat{i} + N\hat{j}$ is defined on some open set containing R, and is C^1. Then

$$\boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\,dx + N\,dy - \sum_{i=1}^{n} \oint_{C_i} M\,dx + N\,dy}$$

(This is the tangential form. The normal form is similar)

<u>Sketch of Proof</u>



For simplicity, only one C_1 inside C

We connect C & C_1 by an "arc" L

and consider the "simple" closed curve

(starting from p): $\quad C^* = C + L - C_1 - L$

Then the region R enclosed between C & C_1 is the region enclosed by C^* except the arc L.

Hence $$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA = \iint_{R\setminus L}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA$$

$$\overset{\text{Green's}}{=} \oint_{C^*} M\,dx+N\,dy$$

$$=\left(\oint_C + \int_L + \oint_{-C_1} + \int_{-L}\right)(M\,dx+N\,dy)$$

$$=\left(\oint_C + \int_L - \oint_{C_1} - \int_L\right)(M\,dx+N\,dy)$$

$$= \oint_C M\,dx+N\,dy - \oint_{C_1} M\,dx+N\,dy$$

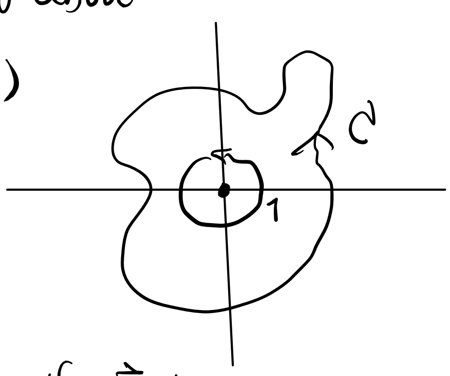
#

eg 49: $\vec{F} = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$ on $\mathbb{R}^2\setminus\{(0,0)\} = \Omega$

we've calculated $\oint_{C_1} \vec{F}\cdot d\vec{r} = 2\pi$ for $C_1: x^2+y^2=1$ (anti-clockwise)

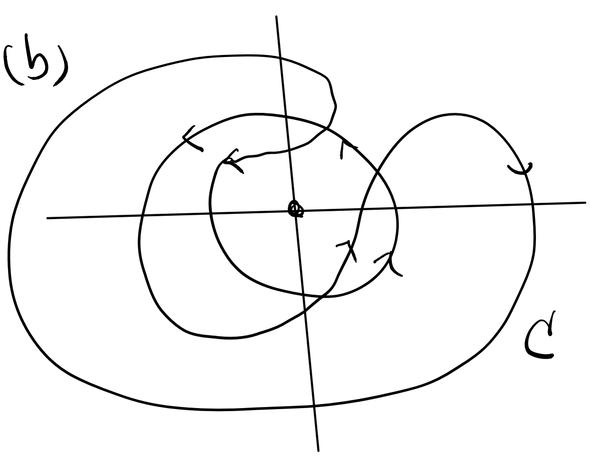
How about

(a)



$\oint_C \vec{F}\cdot d\vec{r} = 2\pi$

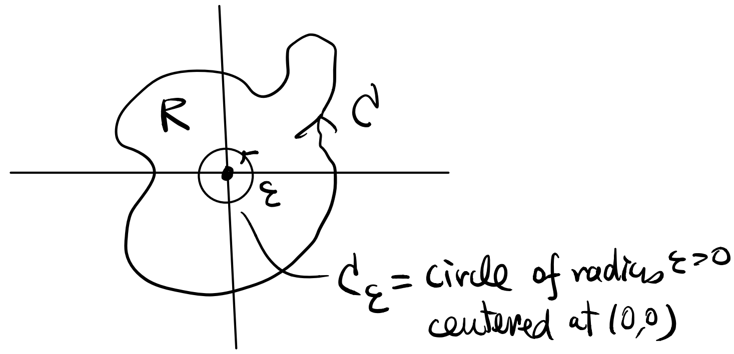
(b)



Soln: (a) Recall that $\vec{\nabla}\times\vec{F}=0$

(Green's Thm doesn't apply to get $\oint_C \vec{F}\cdot d\vec{r}=0$, since C encloses the origin $(0,0)$ where $\vec{F}$ is not defined)

Choose $\varepsilon>0$ small enough such that the circle C_ε of radius ε centered at $(0,0)$ is completely enclosed by C.



$\vec{F}$ is smooth in the region R between C and C_ε.

Hence the general form of Green's Thm applied:

$$0=\iint_R (\vec{\nabla}\times\vec{F})\cdot\hat{k}\,dA=\oint_C \vec{F}\cdot d\vec{r}-\oint_{C_\varepsilon}\vec{F}\cdot d\vec{r}$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r}=\oint_{C_\varepsilon}\vec{F}\cdot d\vec{r}$$

$$=\oint_{C_\varepsilon}\frac{-y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy$$

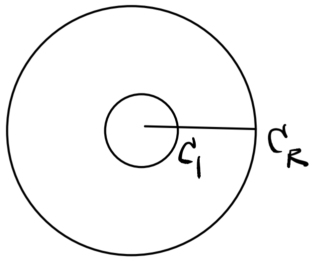
Parametrize C_ε by $\begin{cases} x=\varepsilon\cos\theta \\ y=\varepsilon\sin\theta \end{cases}$, $0\le\theta\le 2\pi$

$$\Rightarrow \oint_{C_\varepsilon}\vec{F}\cdot d\vec{r}=\int_0^{2\pi}\left[\frac{-\varepsilon\sin\theta}{\varepsilon^2}(-\varepsilon\sin\theta)+\frac{\varepsilon\cos\theta}{\varepsilon^2}(\varepsilon\cos\theta)\right]d\theta$$

$$=2\pi$$

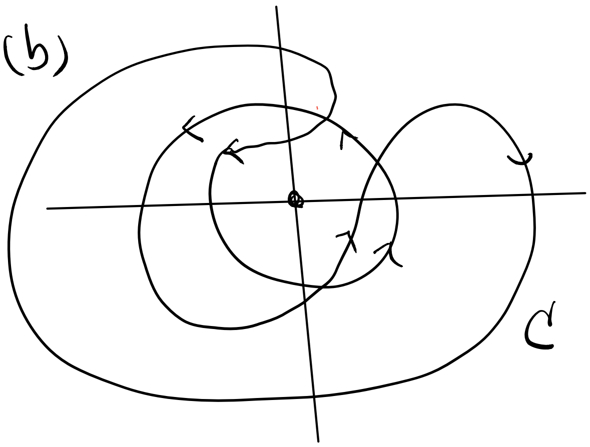
$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r}=2\pi \quad ※$$

(In fact, we've proved that $\oint_{C_R} \vec{F}\cdot d\vec{r} = 2\pi$, $\forall$ radius $R>0$,
which can be seen by consider the domain between C_1 & C_R



Green's Thm
$\Rightarrow \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_R} \vec{F}\cdot d\vec{r}$)

(b)



Decompose the curve into

+

doesn't enclosed (0,0)

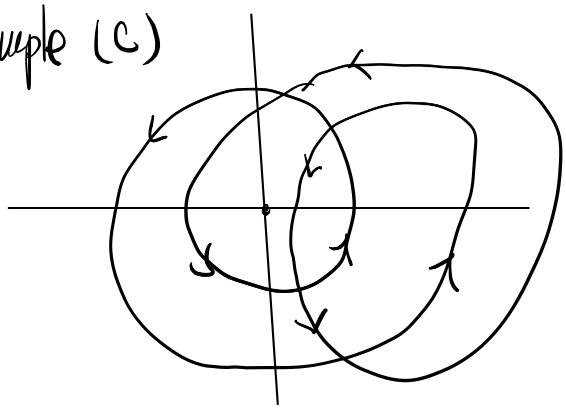
enclosing (0,0) (circle)

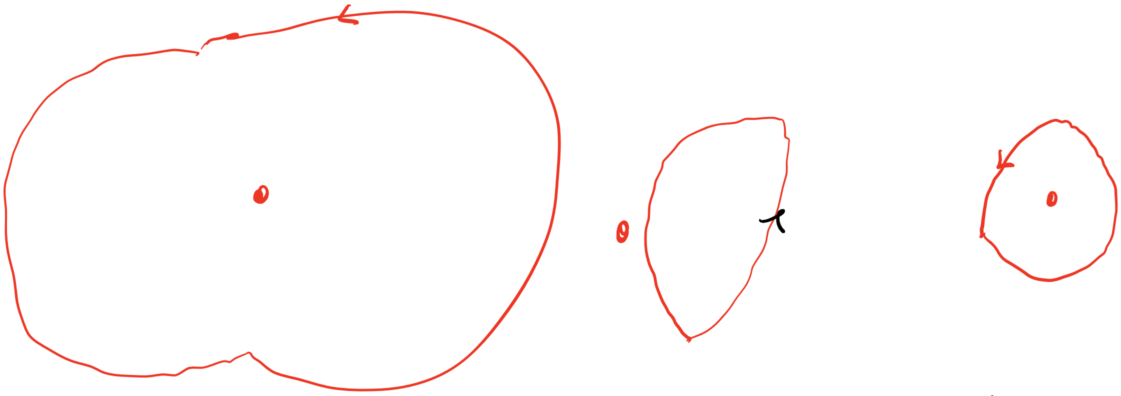
$\Rightarrow \oint_{C_1} \vec{F}\cdot dr = 0$

by part (a) $\oint_{C_2} \vec{F}\cdot dr = 2\pi$

$\therefore \oint_{C} \vec{F}\cdot d\vec{r} = 0 + 2\pi = 2\pi$.

Addition example (c)





$\oint \cdots = 2\pi$ $\qquad$ $\oint \cdots = 0$ $\qquad$ $\oint \cdots = 2\pi$

Hence $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = 2\pi + 0 + 2\pi = 4\pi$

(optional ex! : think of some examples with -2π)

Surface Area & Integral

Def 14 Parametric Surface (Surface with parametrization)

A parametric surface (or a parametrization of a surface) in $\mathbb{R}^3$ is a mapping of 2-variables into $\mathbb{R}^3$:

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

And it is called smooth if

(1) $\vec{r}$ is C^1 (eg. $x_u, x_v, y_u, y_v, z_u, z_v$ are continuous)

(2) $\boxed{\vec{r}_u \times \vec{r}_v \neq 0}$ $\forall\, u, v$

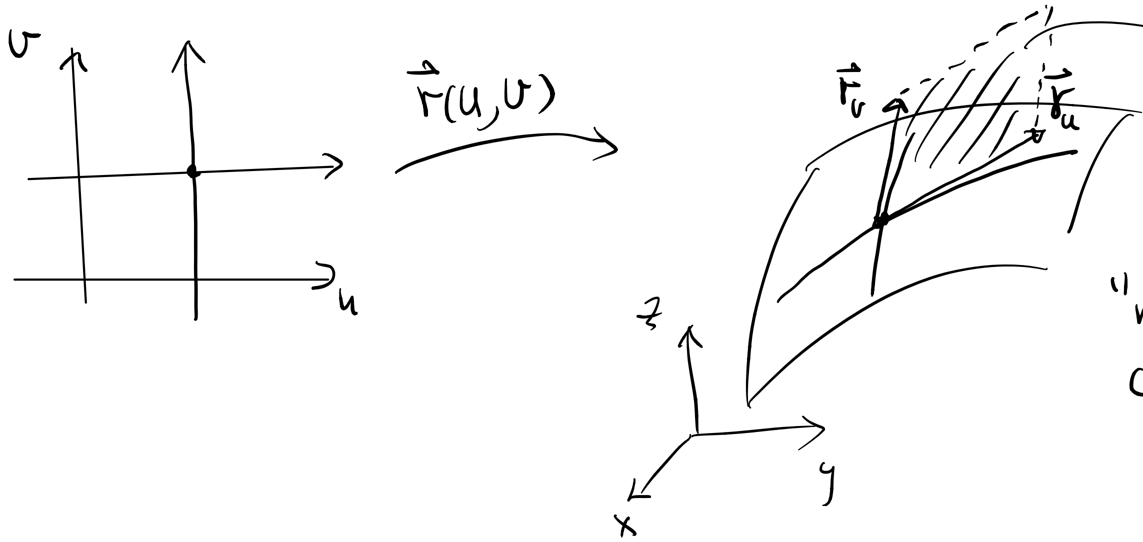
where
$$\begin{cases} \vec{r}_u = \dfrac{\partial \vec{r}}{\partial u} = \dfrac{\partial x}{\partial u}\hat{i} + \dfrac{\partial y}{\partial u}\hat{j} + \dfrac{\partial z}{\partial u}\hat{k} \\ \vec{r}_v = \dfrac{\partial \vec{r}}{\partial v} = \dfrac{\partial x}{\partial v}\hat{i} + \dfrac{\partial y}{\partial v}\hat{j} + \dfrac{\partial z}{\partial v}\hat{k} \end{cases}$$

$$\left(\begin{cases} \vec{r}_u = x_u\hat{i} + y_u\hat{j} + z_u\hat{k} \\ \vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k} \end{cases} \right)$$

Note: Condition (2) $\Rightarrow$ $\vec{r}_u, \vec{r}_v$ are linearly independent

$\Rightarrow$ span$\{\vec{r}_u, \vec{r}_v\}$ is in fact a 2-dim'l subspace.

$\Rightarrow$ "surface" cannot be degenerated to a curve or a point.



area $= |\vec{r}_u \times \vec{r}_v| \neq 0$

Hence, it is a "non-degenerate" condition.