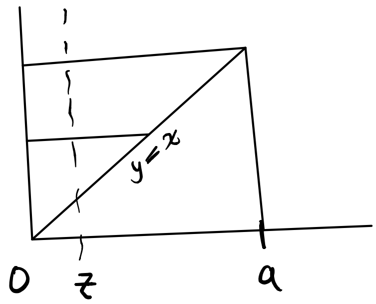
(1)

$$I = \int_0^a \int_0^x \int_0^y f(z)\,dz\,dy\,dx \qquad (a>0)$$

$$\begin{cases} 0 \le x \le a \\ 0 \le y \le x \\ 0 \le z \le y \end{cases}$$



$$\Rightarrow \begin{cases} 0 \le z \le a, \\ z \le y \le x \\ z \le x \le a \end{cases}$$

$$I = \int_0^a \int_z^a \int_z^x f(z)\,dy\,dx\,dz$$

$$= \int_0^a f(z) \left[\int_z^a (x-z)\,dx\right] dz$$

$$= \int_0^a f(z) \left[\frac{x^2}{2} - zx\right]_z^a dz$$

$$= \int_0^a f(z) \left[\left(\frac{a^2}{2} - az\right) - \left(\frac{z^2}{2} - z^2\right)\right] dz$$

$$= \int_0^a f(z) \left[\frac{a^2}{2} - az + \frac{z^2}{2}\right] dz$$

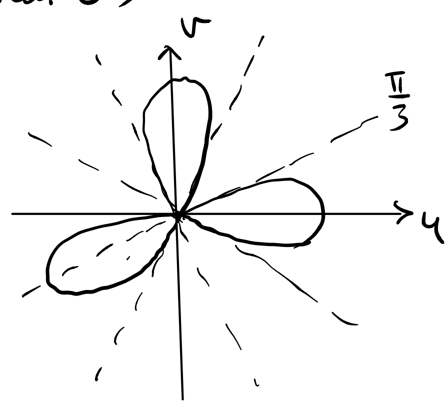
$$= \frac{1}{2}\int_0^a f(z)(a-z)^2\,dz$$

(2) let $u = ax$
$v = by$

$$(u^2+v^2)^2 = u^3 - 3uv^2$$

In polar of $\begin{cases} u = r\cos\theta \\ v = r\sin\theta, \end{cases}$

$$r^4 = r^3(\cos^3\theta - 3\cos\theta\sin^2\theta)$$
$$= r^3\cos 3\theta$$
$$\Rightarrow \quad r = \cos 3\theta$$



$\Rightarrow$ Area bounded by C

$$= \iint_{\substack{(u,v) \\ \text{bdd by} \\ r=\cos 3\theta}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{ab}$$

$$= \frac{1}{ab} \iint_{\substack{(u,v) \\ \text{bdd by} \\ r=\cos 3\theta}} du\,dv$$

$$= \frac{3}{ab} \cdot \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} \cdot r\,dr\,d\theta \qquad \text{by symmetry}$$

$$= \frac{3}{ab} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos^2 3\theta}{2}\, d\theta$$

$\xi = 3\theta$
$d\xi = 3d\theta$

$$= \frac{1}{2ab} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\xi\, d\xi$$

$$= \frac{1}{2ab}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\cos 2\xi+1}{2}\right)d\xi$$

$\cos 2\xi = \cos^2\xi - \sin^2\xi$

$= 2\cos^2\xi - 1$

$$= \frac{1}{2ab}\left[\frac{\sin 2\xi}{4}+\frac{1}{2}\xi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4ab}.$$

(3) $0 \le \iint_R [f(x)-f(y)]^2 dA = \int_a^b \int_a^b (f(x)-f(y))^2 dx\,dy$

$= \int_a^b \int_a^b [f^2(x) - 2f(x)f(y) + f^2(y)]dx\,dy$

$= \int_a^b \left[\left(\int_a^b f^2(x)dx\right) - 2f(y)\int_a^b f(x)dx + f^2(y)(b-a)\right]dy$

$= (b-a)\int_a^b f^2(x)dx - 2\int_a^b f(x)dx \int_a^b f(y)dy$

$+ (b-a)\int_a^b f^2(y)dy$

$= 2(b-a)\int_a^b f^2(x)dx - 2\left(\int_a^b f(x)dx\right)^2$

$\Rightarrow$ $\boxed{\left(\int_a^b f(x)dx\right)^2 \le (b-a)\int_a^b f^2(x)dx \quad \text{for any } f \text{ on } [a,b]}$

(4) $D_{r_0} = \{(x,y) : x^2+y^2 \le r_0^2\}$

$$\Rightarrow \iint_{D_{r_0}} \sin(x^2+y^2)\,dA$$

$$= \int_0^{2\pi}\int_0^{r_0} \sin r^2 \cdot r\,dr\,d\theta$$

$$= \pi \int_0^{r_0^2} \sin t\,dt \quad (\text{sub } t = r^2)$$

$$= \pi \left[-\cos t\right]_0^{r_0^2}$$

$$= \pi\left(1-\cos r_0^2\right)$$

If $r_0^2 = 2n\pi$ i.e. $r_0 = \sqrt{2n\pi}$

then $\displaystyle\iint_{D_{\sqrt{2n\pi}}} \sin(x^2+y^2)\,dA = 0$ for all $n \ge 1$

$$\Rightarrow \lim_{n\to+\infty} \iint_{D_{\sqrt{2n\pi}}} \sin(x^2+y^2)\,dA = 0.$$

However if $r_0^2 = (2n+1)\pi$, i.e. $r_0 = \sqrt{(2n+1)\pi}$

$$\iint_{D_{\sqrt{(2n+1)\pi}}} \sin(x^2+y^2)\,dA = \pi(1-\cos\pi) = 2\pi$$

$$\Rightarrow \lim_{n\to+\infty} \iint_{D_{\sqrt{(2n+1)\pi}}} \sin(x^2+y^2)\,dA = 2\pi$$

Hence $\displaystyle\iint_{\mathbb{R}^2} \sin(x^2+y^2)\,dA$ doesn't make sense as improper integral

5(a) Statement

Let $F : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_2, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{pmatrix}$ near a point p

with $\dfrac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} \neq 0$ at p. Then near a point p,

F can be decomposed into $F = L \circ H \circ K$

with H, K of the forms

$$K : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} k(x_1, x_2, x_3) \\ x_2 \\ x_3 \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad \left(\text{or } \begin{pmatrix} k(x_1, x_2, x_3) \\ x_1 \\ x_3 \end{pmatrix} \text{ or } \begin{pmatrix} k(x_1, x_2, x_3) \\ x_1 \\ x_2 \end{pmatrix} \right)$$

$$H : \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ h(y_1, y_2, y_3) \\ y_3 \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad \left(\text{or } \begin{pmatrix} y_1 \\ h(y_1, y_2, y_3) \\ y_2 \end{pmatrix} \right)$$

$$L : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ l(z_1, z_2, z_3) \end{pmatrix}$$

such that $\det DK \neq 0$, $\det D(H) \neq 0$, & $\det D(L) \neq 0$ at the corresponding pt.

Pf By assumption $0 \neq \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)}$ near p

$\Rightarrow$ At least one of $\frac{\partial f}{\partial x_i}(p) \neq 0$, $i = 1, 2, 3$.

Case 1 $\frac{\partial f}{\partial x_1}(p) \neq 0$

Then define

$$K: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 = f_1(x_1, x_2, x_3) \\ y_2 = x_2 \\ y_3 = x_3 \end{pmatrix}$$ is of the required form (i.e. $k(x_1, x_2, x_3) = f_1(x_1, x_2, x_3)$)

and

$$J(K) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \det J(K)(p) = \frac{\partial f_1}{\partial x_1}(p) \neq 0$$

By Inverse Function Theorem, K is invertible near p

and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = K^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} g(y_1, y_2, y_3) \\ x_2 \\ x_3 \end{pmatrix}$$

is differentiable at $K(p)$ (since $y_2 = x_2$, $y_3 = x_3$)

[Note: $x_1 = g(y_1, y_2, y_3) = g(f_1(x_1, x_2, x_3), x_2, x_3)$]

By $D(K^{-1})_{K(p)} \cdot DK_p = Id$

$$\begin{pmatrix} \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} & \frac{\partial g}{\partial y_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_1} & \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_2}+\frac{\partial g}{\partial y_2} & \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_3}+\frac{\partial g}{\partial y_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_1} = 1 \\ \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_2}+\frac{\partial g}{\partial y_2} = 0 \\ \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_3}+\frac{\partial g}{\partial y_3} = 0 \end{cases}$$

In particular $\det D(K^{-1})_{K(p)} = \dfrac{1}{\det D(K)_p} \neq 0$

Define

$$(*)\begin{cases} m(y_1, y_2, y_3) = f_2(g(y_1,y_2,y_3), y_2, y_3) \\ n(y_1, y_2, y_3) = f_3(g(y_1,y_2,y_3), y_2, y_3) \end{cases}$$

Then putting $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = K\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1(x_1,x_2,x_3) \\ x_2 \\ x_3 \end{pmatrix}$ into $(*)$

$$(*)_2\begin{cases} m(f_1(x_1,x_2,x_3), x_2, x_3) = f_2(g(f_1(x_1,x_2,x_3), x_2, x_3), x_2, x_3) \\ \qquad = f_2(x_1, x_2, x_3) \\ n(f_1(x_1,x_2,x_3), x_2, x_3) = f_3(g(f_1(x_1,x_2,x_3), x_2,x_3), x_2, x_3) \\ \qquad = f_3(x_1, x_2\ x_3) \end{cases}$$

(since $g(f_1(x_1,x_2,x_3), x_2, x_3) = x_1$)

Now define

$$M : \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ m(y_1, y_2, y_3) \\ n(y_1, y_2, y_3) \end{pmatrix}$$

Then

$$M \circ K \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = M \begin{pmatrix} f_1(x_1, x_2, x_3) \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} f_1(x_1, x_2, x_3) \\ m(f_1(x_1, x_2, x_3), x_2, x_3) \\ n(f_1(x_1, x_2, x_3), x_2, x_3) \end{pmatrix}$$

$$(\text{by } (*)_2) \quad = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{pmatrix} = F \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\therefore \; F = M \circ K.$

By definition of M,

$$DM = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial m}{\partial y_1} & \frac{\partial m}{\partial y_2} & \frac{\partial m}{\partial y_3} \\ \frac{\partial m}{\partial y_1} & \frac{\partial m}{\partial y_2} & \frac{\partial m}{\partial y_3} \end{pmatrix}$$

$$\Rightarrow \det DM = \det \begin{pmatrix} \frac{\partial m}{\partial y_2} & \frac{\partial m}{\partial y_3} \\ \frac{\partial n}{\partial y_2} & \frac{\partial n}{\partial y_3} \end{pmatrix}$$

By Chain rule $\quad DF = DM \circ DK$

$$\Rightarrow \quad \det DF_p = \det DM_{K(p)} \cdot \det DK_p$$

$$\Rightarrow \quad \det DM_{K(p)} \neq 0 \qquad \text{since } \det DF_p \neq 0, \ \det DK_p \neq 0$$

$$\Rightarrow \quad \det \begin{pmatrix} \frac{\partial m}{\partial y_2} & \frac{\partial m}{\partial y_3} \\ \frac{\partial n}{\partial y_2} & \frac{\partial n}{\partial y_3} \end{pmatrix} \neq 0 \quad \text{at the pt } y_0 = K(p)$$

Now for a fixed y_1,

consider $\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \mapsto \begin{pmatrix} m(y_1, y_2, y_3) \\ n(y_1, y_2, y_3) \end{pmatrix}$

as 2-variables to 2-variables transformation

with $\frac{\partial(m,n)}{\partial(y_2,y_3)} \neq 0$ at the pt. $K(p)$.

Then <u>Step 1 in the proof of the 2-dim</u>

$\Rightarrow$ The maps $\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \mapsto \begin{pmatrix} m(y_1, y_2, y_3) \\ n(y_1, y_2, y_3) \end{pmatrix}$

decomposes to

$M = L \circ H$ with H & L of the form

$$H : \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ h(y_1, y_2, y_3) \\ y_3 \end{pmatrix} \left(\begin{pmatrix} \\ y_1, y_2, y_3) \\ y_2 \end{pmatrix} \right)$$

and $L: \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} z_1 \\ z_2 \\ l(z_1, z_2, z_3) \end{pmatrix}$

with non-zero Jacobian determinants at the corresponding points: $DH_{K(p)} \neq 0$, $DL_{H\circ K(p)} \neq 0$.

Note that for each fixed y_1, the diffeomorphism

$$\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \mapsto \begin{pmatrix} m(y_1, y_2, y_3) \\ n(y_1, y_2, y_3) \end{pmatrix} \text{ depends on } y_1$$

and hence the decomposition, that is functions

$h(y_1, y_2, y_3)$ and $l(z_1, z_2, z_3) = l(y_1, z_2, z_3)$

(since $z_1 = y_1$)

are also depend on y_1, and can be considered as functions of 3-variables.

Then $F = M \circ K$

$= L \circ H \circ K$

with the required form and non-zero Jacobian determinants at corresponding.

Since $DF_p \neq 0$, $\left(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \frac{\partial f_1}{\partial x_3}\right) \neq (0,0,0)$.

Hence, if $\frac{\partial f_1}{\partial x_1} = 0$, then either $\frac{\partial f_1}{\partial x_2}$ or $\frac{\partial f_1}{\partial x_3} \neq 0$

then interchanging coordinates, applying the above argument, and changing back. We obtained the other situation for K. ※

Remark: Many of you try to go directly to $H = \begin{pmatrix} y_1 \\ G(y_1, y_2, y_3) \\ y_3 \end{pmatrix}$,

which cannot be done by just the assumption $\frac{\partial f_1}{\partial x_1}(p) \neq 0$, one needs $\det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}(p) \neq 0$ too. In doing so, the correct and "clear" argument is: $\det F \neq 0$

$\Rightarrow$ at least one of $\det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$, $\det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_3} \end{pmatrix}$

$\det\begin{pmatrix} \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{pmatrix}$ is nonzero. Then may take

$\det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \neq 0$ as case 1. And within case 1, there are 2 subcases, $\frac{\partial f_1}{\partial x_1} \neq 0$ or $\frac{\partial f_1}{\partial x_2} \neq 0$.

Also, many of you forgot to stay the alternative possibility.

step2 : let $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = K\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} k(x_1, x_2, x_3) \\ x_2 \\ x_3 \end{pmatrix}$

be a diffeomaphism from region R_1 to $R_2 = K(R_1)$.

Then for any function $f(y_1, y_2, y_3)$ on R_2,

$$\iiint_{R_2} f(y_1, y_2, y_3)\, dy_1 dy_2 dy_3$$

$$= \iiint_{R_1} f(k(x_1,x_2,x_3), x_2, x_3) \left| \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} \right| dx_1 dx_2 dx_3$$

Pf : By additivity property of integrations and cutting R_1 (and correspondingly $R_2 = K(R_1)$) into small regions we may assume

$$R_1 = [a,b] \times [c,d] \times [r,s]$$
$$= \{ a \le x_1 \le b,\ c \le x_2 \le d,\ r \le x_3 \le s \}$$

For any fixed $(y_2, y_3) = (x_2, x_3)$

$$y_1 = k(x_1, x_2, x_3) = k(x_1, y_2, y_3),$$
$$(\text{for } a \le x_1 \le b,)$$

can be regarded as a transformation of 1-variable

Note that $\dfrac{\partial y_1}{\partial x_1} = \dfrac{\partial k}{\partial x_1} = \det \begin{pmatrix} \frac{\partial k}{\partial x_1} & \frac{\partial k}{\partial x_2} & \frac{\partial k}{\partial x_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det D(K) \neq 0$

(K : diffeo)

Note also that R_2 is of special type

$$\{ c \le y_2 \le d,\ r \le y_3 \le s,\ k(a, y_2, y_3) \le y_1 \le k(b, y_2, y_3) \}$$

$\left(\text{if } \frac{\partial y_1}{\partial x_1} > 0 \right)$

or

$$\{ c \le y_2 \le d,\ r \le y_3 \le s,\ k(b, y_2, y_3) \le y_1 \le k(a, y_2, y_3) \}$$

$\left(\text{if } \frac{\partial y_1}{\partial x_1} < 0 \right)$

By Fubini's Thm (assuming $\frac{\partial y_1}{\partial x_1} > 0$, the other case is similar)

$$\iiint_{R_2} f(y_1, y_2, y_3)\, dy_1\, dy_2\, dy_3 = \int_r^s \int_c^d \left[\int_{k(a, y_2, y_3)}^{k(b, y_2, y_3)} f(y_1, y_2, y_3)\, dy_1 \right] dy_2\, dy_3$$

(change of variable formula in 1-dim)

$$= \int_r^s \int_c^d \left[\int_a^b f(k(x_1, y_2, y_3), y_2, y_3) \frac{\partial y}{\partial x_1} dx_1 \right] dy_2\, dy_3$$

$$= \int_r^s \int_c^d \int_a^b f(k(x_1, x_2, x_3), x_2, x_3) \left| \det D(K) \right| dx_1\, dx_2\, dx_3$$

$\left(\text{since } \frac{\partial y_1}{\partial x_1} > 0 \right)$

$$= \iiint_{R_1} f(k_1(x_1, x_2, x_3), x_2, x_3) \left| \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} \right| dx_1\, dx_2\, dx_3$$

$\#$