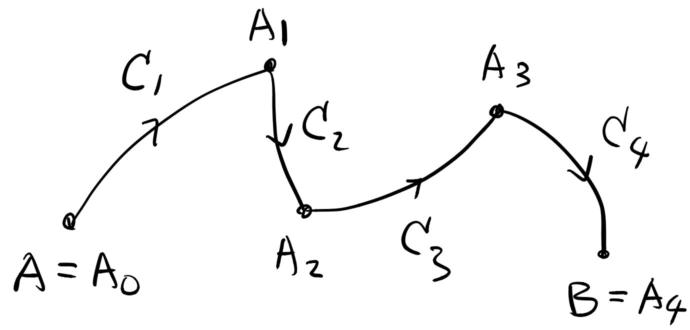
Part 2 : For a general piecewise smooth curve

$$C = C_1 \cup C_2 \cup \cdots \cup C_k$$



($= C_1 + C_2 + \cdots + C_k$ in order to indicate that the orientation of C_i, $i=1,\dots,k$, are correct wrt the orientation of C)

where C_i is smooth going from A_{i-1} to A_i.

Then part 1 implies

$$\int_C \vec{F} \cdot \hat{T}\, ds = \sum_{i=1}^{k} \int_{C_i} \vec{F} \cdot \hat{T}\, ds$$

$$= \sum_{i=1}^{k} \left[f(A_i) - f(A_{i-1}) \right] \quad \text{(by part 1)}$$

$$= f(A_k) - f(A_0)$$

$$= f(B) - f(A)$$

#

Thm 9 Let $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, be open and connected.

$\vec{F}$ is a continuous vector field on Ω. Then the following are equivalent.

(a) $\exists$ a C^1 function $f: \Omega \to \mathbb{R}$ such that

$$\vec{F} = \vec{\nabla} f$$

(b) $\oint_C \vec{F} \cdot d\vec{r} = 0$ along any closed curve C on Ω.

(c) $\vec{F}$ is conservative.

Pf: "(a) $\Rightarrow$ (b)"

If f is C^1 and $\vec{F} = \vec{\nabla} f$

and $\vec{r}: [a,b] \to \Omega$ parametrizes C (any closed curve)

C closed $\Rightarrow$ $\vec{r}(a) = \vec{r}(b) = A$

Fundamental Thm of Line Integral $\Rightarrow$

$$\oint_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

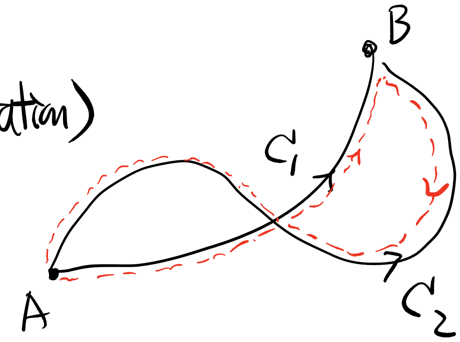
$$= f(A) - f(A) = 0.$$

"(b) $\Rightarrow$ (c)" Suppose C_1, C_2 are oriented curves with starting point A and end point B.

Then $C_1 \cup (-C_2)$

$= C_1 - C_2$ (a better notation)

is an oriented closed curve.



Then by (b)

$$0 = \oint_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{-C_2} \vec{F} \cdot d\vec{r}$$

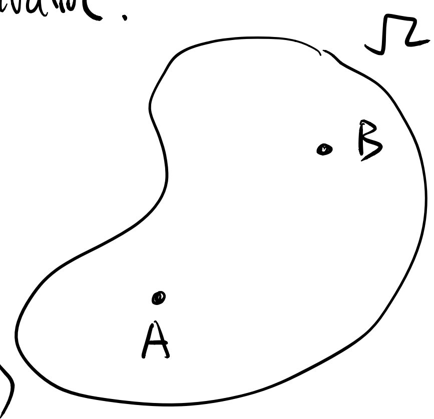
$$= \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r}$$

Since C_1 & C_2 are arbitrary, $\vec{F}$ is conservative.

" (c) $\Rightarrow$ (a) "

Assume $n=2$ for simplicity (other dimensions are similar)

Let $\vec{F} = M\hat{i} + N\hat{j}$ are conservative.



Fix a point $A \in \Omega$

Then for any point $B \in \Omega$, define

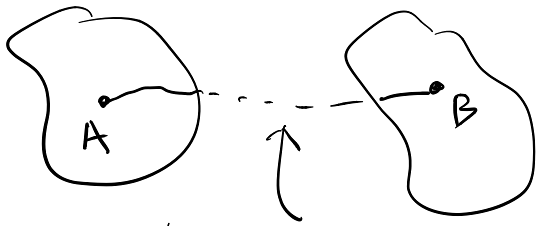
$$f(B) = \int_A^B \vec{F} \cdot \hat{T}\, ds = \underbrace{\text{common value}}_{} \text{ of } \int_C \vec{F} \cdot \hat{T}\, ds$$

($\vec{F}$ is conservative)

for any C from A to B.

Since $\vec{F}$ is conservative, $f(B)$ is well-defined.

[We've also used the assumption that Ω is connected, otherwise there is no path from A to B, if A, B belong to different connected components:

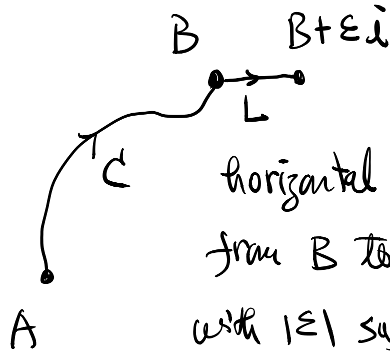


]

Claim $\vec{F} = \vec{\nabla} f$

Pf of claim: $\frac{\partial f}{\partial x}(B) = \lim_{\varepsilon \to 0} \frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon}$

Let C be an oriented curve from A to B.



horizontal line segment from B to $B+\varepsilon i$ with $|\varepsilon|$ sufficiently small s.t. $C + L \subset \Omega$ open

Then $f(B+\varepsilon\hat{i})$

$$= \int_A^{B+\varepsilon\hat{i}} \vec{F} \cdot d\vec{r}$$

$$= \int_{C+L} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_L \vec{F} \cdot d\vec{r}$$

$$= \int_A^B \vec{F} \cdot d\vec{r} + \int_L \vec{F} \cdot d\vec{r}$$

$$= f(B) + \int_L \vec{F} \cdot d\vec{r}$$

$$\therefore \quad \frac{f(B+\varepsilon \hat{i}) - f(B)}{\varepsilon} = \frac{1}{\varepsilon}\int_L \vec{F}\cdot d\vec{r} \qquad \left(\begin{array}{c} \text{parametrize } L \text{ by} \\ B + t\hat{i},\ t\in[0,\varepsilon] \\ B=(x,y) \end{array} \right)$$

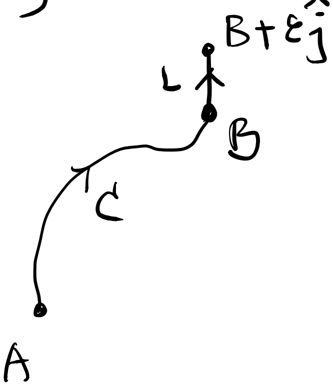
$$= \frac{1}{\varepsilon}\int_0^{\varepsilon} M(x+t,y)\,dt$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}(B) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^{\varepsilon} M(x+t,y)\,dt$$

$$= M(x,y) \quad (\text{by MVF } \& \ M \text{ is continuous})$$

$(\vec{F}$ is continuous$)$

Similarly $\frac{\partial f}{\partial y}(B) = N(x,y)$ by consider:



(vertical line segment)

So $\vec{\nabla} f = \vec{F}$

Since $\vec{F}$ is continuous,

$M = \frac{\partial f}{\partial x}$ & $N = \frac{\partial f}{\partial y}$ are continuous

$\Rightarrow f \in C^1$ ✗✗

Remark: The function f in (a) of Thm 9 is called the potential function of $\vec{F}$. It is unique up to an additive constant:

$$\vec{\nabla}(f+c) = \vec{F}, \quad \forall \text{ const. } c.$$

Corollary (to Thm 9)

Let $\vec{F}$ be conservative and C^1

connected open

"n=3" If $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ (on $\Omega \subset \mathbb{R}^3$)

then
$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \\[2mm] \dfrac{\partial N}{\partial z} = \dfrac{\partial L}{\partial y} \\[2mm] \dfrac{\partial L}{\partial x} = \dfrac{\partial M}{\partial z} \end{cases}$$

connected open

"n=2" If $\vec{F} = M\hat{i} + N\hat{j}$ (on $\Omega \subset \mathbb{R}^2$)

then
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Pf: $\vec{F}$ conservative $\xRightarrow{\text{Thm 9}}$ $\vec{F} = \vec{\nabla} f$ for some function f.

i.e. $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$
$= M\hat{i} + N\hat{j} + L\hat{k}$

$\vec{F} \in C^1 \Rightarrow f \in C^2$

Hence mixed derivatives thm (Clairaut's Thm)

$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial N}{\partial x} \\[2mm] \dfrac{\partial N}{\partial z} = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial L}{\partial y} \\[2mm] \dfrac{\partial L}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial M}{\partial z} \end{cases}$$

(included "n=2" case)

#

eg 42: Show that $\vec{F}(x,y) = \hat{i} + x\hat{j}$ is not conservative in $\mathbb{R}^2$.

Soln: ($\vec{F} \in C^\infty$)

$$\begin{cases} M \equiv 1 \\ N = x \end{cases} \Rightarrow \begin{cases} \dfrac{\partial M}{\partial y} = 0 \\ \dfrac{\partial N}{\partial x} = 1 \end{cases} \neq$$

By Cor to Thm 9, $\vec{F}$ is not conservative. ##

Remark (Important)

For a C^1 vector field $\vec{F} = M\vec{i} + N\vec{j} + L\vec{k}$

$$\boxed{\vec{F} \text{ conservative}} \;\; \underset{?}{\overset{\text{Cor to Thm 9}}{\rightleftarrows}} \;\; \boxed{\begin{array}{c} M, N, L \text{ satisfy the system} \\ \text{of PDEs in the Cor to Thm 9} \end{array}}$$

Answer: NOT TRUE in general, needs extra condition on the domain Ω ("connected" is not enough)

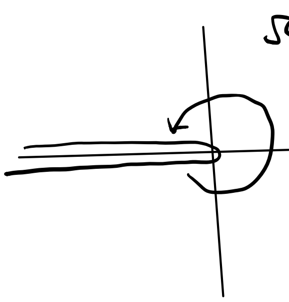
---

eg 43 Consider the vector field

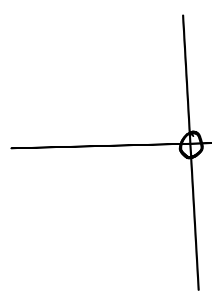
$$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$$

and the domains $\Omega_1 = \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \leq 0\}$

$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$

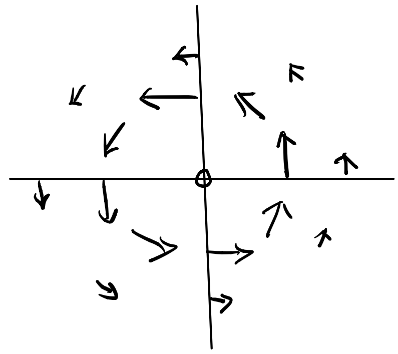


Ω_1 doesn't include the negative x-axis (and origin)



Ω_2 doesn't include the origin (but include the negative x-axis)

In polar coordinates



$$\vec{F} = -\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j}$$

$\Rightarrow$
- $\vec{F}$ rotates around the origin anti-clockwisely
- $|\vec{F}| = \frac{1}{r} \to 0$ as $r \to +\infty$
- $|\vec{F}| = \frac{1}{r} \to +\infty$ as $r \to 0$ $\Rightarrow$ $\vec{F}$ cannot be extended to a C^1 vector field on $\mathbb{R}^2$.

Besides $(0,0)$, $\vec{F}$ is C^1 and

$\vec{F}$ is C^1 on Ω_1, and also

$\vec{F}$ is C^1 on Ω_2.

Questions: Is $\vec{F}$ conservative on Ω_1?

Is $\vec{F}$ conservative on Ω_2?

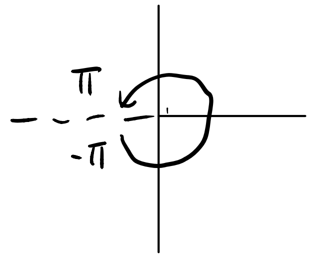
Soln: (1) For Ω_1, and (x,y) can be expressed in polar coordinates with

$$\begin{cases} r > 0 \\ -\pi < \theta < \pi \end{cases}$$ (r,θ) are unique

doesn't include end values.

Define $f(x,y) = \theta$ "smooth" on Ω_1 (check!)

Then $\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial \theta}{\partial x} = -\dfrac{\sin\theta}{r} \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{\partial \theta}{\partial y} = \dfrac{\cos\theta}{r} \end{cases}$

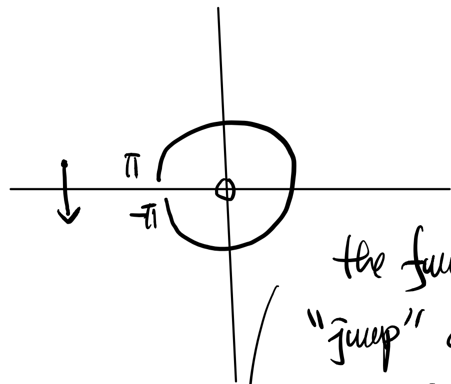


$$\Rightarrow \quad \vec{F} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \vec{\nabla} f$$

$\Rightarrow$ $\vec{F}$ is conservative (by Thm 9)

(2) For Ω_2, the function $f(x,y)=\theta$ cannot be extended to a "smooth" function on (the whole) Ω_2

$\therefore$ $f(x,y)=\theta$ doesn't work in the case of Ω_2.



(the function $f=\theta$ "jump" at the negative x-axis
$\Rightarrow$ f cannot be extended to a continuous function across the $-x$-axis.)

To show that $\vec{F}$ is not conservative on Ω_2, we consider a closed curve

$$C: \ \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \quad t \in [-\pi, \pi]$$

(unit circle in Ω_2, but it is not a curve in Ω_1)

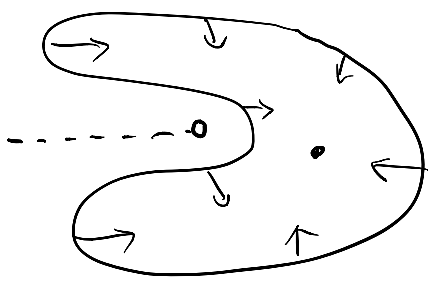
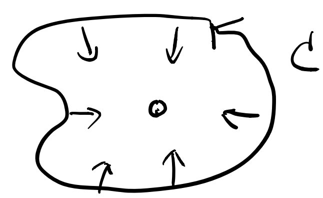
Then
$$\oint_C \vec{F}\cdot d\vec{r} = \int_{-\pi}^{\pi} \left(-\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j}\right)\cdot \vec{r}\,'(t)\,dt$$
$$= \int_{-\pi}^{\pi} (-\sin t\,\hat{i} + \cos t\,\hat{j})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j})\,dt$$
$$= \int_{-\pi}^{\pi} 1\,dt$$

$$= 2\pi$$

$$\neq 0$$

By Thm 9, $\vec{F}$ is not conservative on Ω_2 #

Summary

Ω_1	Ω_2
$f(x,y)=\theta$ smooth function on Ω_1	$f(x,y)=\theta$ is <u>not</u> a smooth function on Ω_2 (θ cannot be well-defined on the whole Ω_2)
$C: x^2+y^2=1$ is <u>not</u> a curve in Ω_1 because: $(-1,0)\in C$ but $(-1,0)\notin \Omega_1$	$C: x^2+y^2=1$ is a closed curve in Ω_2
 closed curve <u>cannot</u> circle around the origin $\Rightarrow$ closed curves can be deformed continuously (within Ω_1) to a point (in Ω_1)	 C encloses the "hole" $\Rightarrow$ C cannot be deformed continuously (within Ω_2) to a point (in Ω_2)

Def 15 A subset $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is called simply-connected if every closed curve in Ω can be contracted to a point in Ω without ever leaving Ω.

(contracted : deformed continuously)

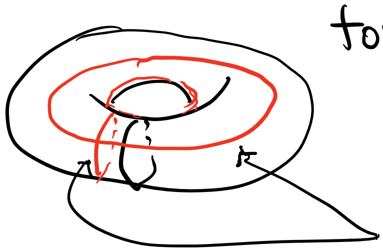
eg 44 Ω_1 in eg 43 is simply-connected, but Ω_2 is not simply-connected.

eg 45:

$S^2 \subset \mathbb{R}^3$ $\quad S^2 = \{x^2+y^2+z^2=1\}$

is simply-connected.

eg 46:



torus $\mathbb{T}^2 \cong S^1 \times S^1 \subset \mathbb{R}^3$

is not simply-connected

these 2 closed curves cannot be contracted to a point on $\mathbb{T}^2$.

Remark: Simply-connectedness is a global condition to guarantee " "PDEs in Co to Thm 9" $\Rightarrow$ "conservative" "

Thm 10: Suppose $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, is connected and simply-connected. Let $\vec{F}$ be C^1 vector field on Ω.

Then

$\vec{F}$ is conservative on Ω $\iff$ components of $\vec{F}$ satisfy the system of PDEs in the Cor to Thm 9.

(Pf: later)

eg 47: Let $\Omega \equiv \mathbb{R}^3$ (connected and simply-connected)

Let
$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} = (y+e^z)\hat{i} + (x+1)\hat{j} + (1+xe^z)\hat{k}.$$

Find the potential function f of $\vec{F}$, i.e.

$$\vec{\nabla} f = \vec{F}$$

Soln: This is, we want to solve

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = L.$$

Checking M, N, L satisfy the system of PDEs in Cor to the Thm 9:

$$\frac{\partial M}{\partial x} = 0 \qquad \frac{\partial M}{\partial y} = 1 \qquad \frac{\partial M}{\partial z} = e^z$$

$$\frac{\partial N}{\partial x} = 1 \qquad \frac{\partial N}{\partial y} = 0 \qquad \frac{\partial N}{\partial z} = 0$$

$$\frac{\partial L}{\partial x} = e^z \qquad \frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial z} = xe^z$$

Thm 10 $\Rightarrow$ existence of potential function f.