eg 40: Let $\vec{F} = (x-y)\hat{i} + x\hat{j}$

$C: x^2+y^2=1$

Find the flow (anti-clockwisely) along C and flux across C.

Soln: Let $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$

(note: correct orientation) (check!)

Then flow $= \oint_C \vec{F}\cdot\hat{T}\,ds = \oint_C \vec{F}\cdot d\vec{r}$

$$= \int_0^{2\pi} \left[(\cos t - \sin t)\hat{i} + \cos t\,\hat{j}\right]\cdot\left[-\sin t\,\hat{i} + \cos t\,\hat{j}\right] dt$$

$$= \cdots = 2\pi \quad \text{(check!)}$$

flux $= \oint_C \vec{F}\cdot\hat{n}\,ds$

$$= \oint_C M\,dy - N\,dx \quad \text{(with anti-clockwise parametrization)}$$

$$= \int_0^{2\pi} (\cos t - \sin t)\,d(\sin t) - \cos t\,d(\cos t)$$

$$= \int_0^{2\pi} \left[(\cos t - \sin t)\cos t + \cos t \sin t\right] dt$$

$$= \cdots = \pi \quad \text{(check!)}$$

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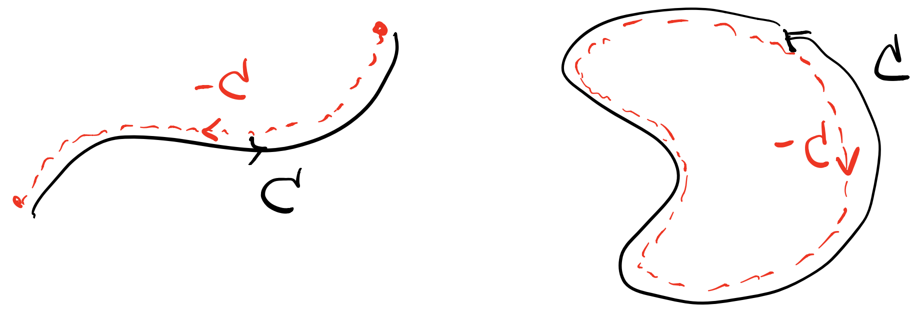
Remark: If C is an oriented curve, then denote by "$-C$" the oriented curve with opposite orientation



- If f is a scalar function

$$\boxed{\int_C f\,ds = \int_{-C} f\,ds}$$

as "ds" is not oriented, just "length"

- If $\vec{F}$ is a vector field

flow

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = -\int_{-C} \vec{F}\cdot\hat{T}\,ds}$$

this $\hat{T}$ is the "$\hat{T}$ for $-C$"

More precise formula:

$$\boxed{\int_C \vec{F}\cdot\hat{T}_C\,ds = -\int_{-C} \vec{F}\cdot\hat{T}_{-C}\,ds}$$

- But for flux

$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \oint_{-C} \vec{F}\cdot\hat{n}\,ds}$$

$\hat{n}$ always outward.

Summary:

scalar f	$\int_C f\,ds$ indep. of orientation	f, ds have no direction
vector $\vec{F}$ flow	$\int_C \vec{F}\cdot\hat{T}\,ds$ depends on orientation	$\hat{T}$ depends on direction
flux	$\int_C \vec{F}\cdot\hat{n}\,ds$ indep. of orientation	$\hat{n}$ always outward

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Conservation Vector Field

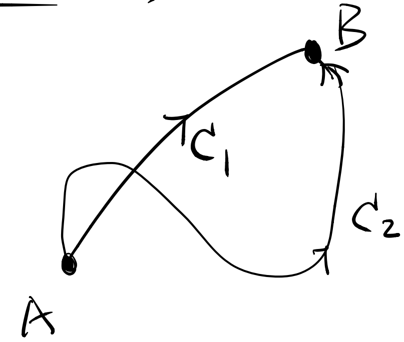
Def 14: Let $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, be open. A vector field $\vec{F}$ defined on Ω is said to be conservative if

$$\int_C \vec{F}\cdot\hat{T}\,ds \left(= \int_C \vec{F}\cdot d\vec{r}\right)$$ along an oriented curve C in Ω

depends only on the starting point and end point of C.

Note: This is usually referred as "path independent".

i.e. If C_1 & C_2 are oriented curves with the same starting point A and end point B,



then

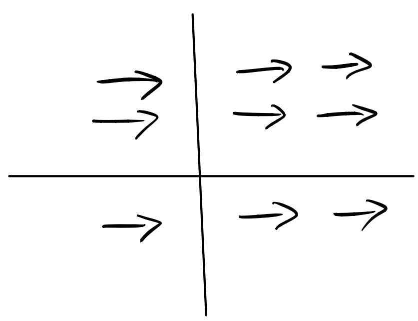
$$\int_{C_1} \vec{F}\cdot\hat{T}\,ds = \int_{C_2} \vec{F}\cdot\hat{T}\,ds$$

(so the value only depends on the points A & B (& direction))

Notation: If $\vec{F}$ is conservative, we sometimes write

$$\boxed{\int_A^B \vec{F}\cdot\hat{T}\,ds}$$ to denote the common value of $\int_C \vec{F}\cdot\hat{T}\,ds$ along any oriented curve C from A to B.

eg 41: $\vec{F} \equiv \hat{i}$ on $\mathbb{R}^2$



$C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$a \le t \le b$

Then $\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$

$$= \int_a^b x'(t)\,dt = x(b) - x(a)$$

(x-coordinate at $\vec{r}(b)$ & $\vec{r}(a)$)

$\therefore$ $\int_C \vec{F}\cdot\hat{T}\,ds$ depends only on the starting point and end point.

$\Rightarrow$ $\vec{F}$ is conservative.

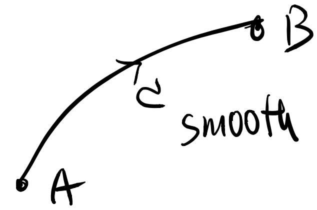
(Note: $\vec{F} = \vec{\nabla} f$ where $f(x,y) = x$)

Thm 8 (Fundamental Theorem of Line Integral)

Let f be a C^1 function on an open set $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, and $\vec{F} = \vec{\nabla} f$ be the gradient vector field of f. Then for any piecewise smooth oriented curve C on Ω with starting point A and end point B,

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = f(B) - f(A)}$$

Pf: Part 1 Assume C is a smooth curve parametrized by $\vec{r}(t)$, $a \le t \le b$



Then $\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

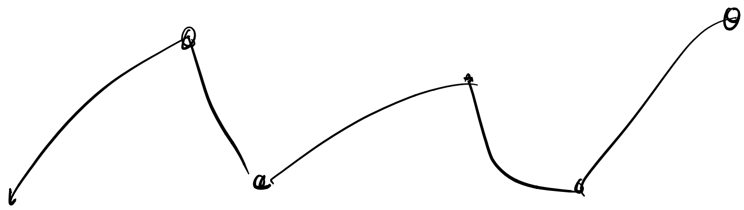
$$= \int_a^b \vec{\nabla} f(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\big(f(\vec{r}(t))\big)\,dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$ (1-variable fund. thm. of Calculus)

$$= f(B) - f(A)$$

(next time)
Part 2



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