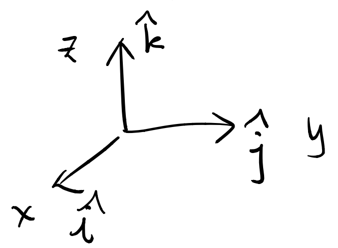
# Vector Analysis

**Notation**: Usually in textbooks, vectors are denoted by boldface $\mathbf{i}$, but hard to do it on screen, so my notation of vectors are:

$$\begin{cases} \text{general vectors} : & \vec{V},\ \vec{F},\ \vec{r},\ \vec{\nabla},\ \ldots \quad (\text{differential operator}) \\ \text{unit vectors} : & \hat{i},\ \hat{j},\ \hat{k},\ \hat{n},\ \hat{T},\ \ldots \end{cases}$$




## Line integrals in $\mathbb{R}^3$ ($\mathbb{R}^n$)

(path integrals)

**Def 9**: The line integral of a function $f$ on a curve (path, line) $C$ with parametrization

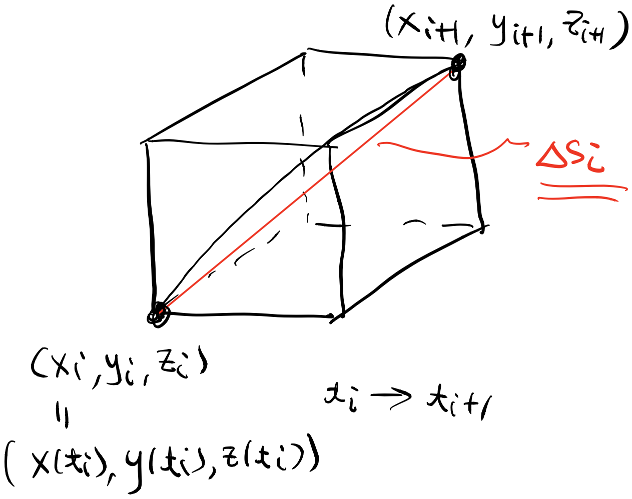
$$\begin{array}{cccc} \vec{r}: & [a,b] & \longrightarrow & \mathbb{R}^3 \\ & \psi & & \psi \\ (\text{position vector}) & t & \longmapsto & (x(t), y(t), z(t)) \end{array}$$

is $\displaystyle\int_C f(\vec{r})\, ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\vec{r}(t_i))\, \Delta s_i$

where $P$ is a partition of $[a,b]$, and

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

i.e. $ds$ = length element of the curve




Remarks:

(1) If $f \equiv 1$, $\boxed{\int_C ds = \text{arc-length of } C}$

(2) The definition is well-defined, i.e. the RHS in the definition is independent of the parametrization $\vec{r}(t)$.

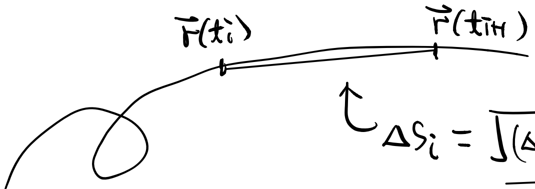
**Def 9'** (Formula for line integral)

Notations as in Def 9, then

$$\int_C f(\vec{r})\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

where $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

Since




$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

$$= \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2}\ \Delta t_i$$

$$\simeq \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\ \Delta t_i$$

$$= |\vec{r}\,'(t_i)|\ \Delta t_i$$

Remarks (1) "$ds = |\vec{r}'(t)|\,dt$" is usually referred as the arc-length element, where $\vec{r}'(t) = (x'(t), y'(t), z'(t))$,

and $|\vec{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.

(2) Suppose the curve $C$ is parametrized by a new parameter $\tilde{t}$

$$\begin{array}{ccc} t & \longleftrightarrow & \tilde{t} \\ \cap & & \cap \\ [a,b] & & [\tilde{a}, \tilde{b}] \end{array} \quad \left( t \leftrightarrow \tilde{t} \text{ is increasing} \quad \frac{d\tilde{t}}{dt} > 0,\ \frac{dt}{d\tilde{t}} > 0 \right).$$

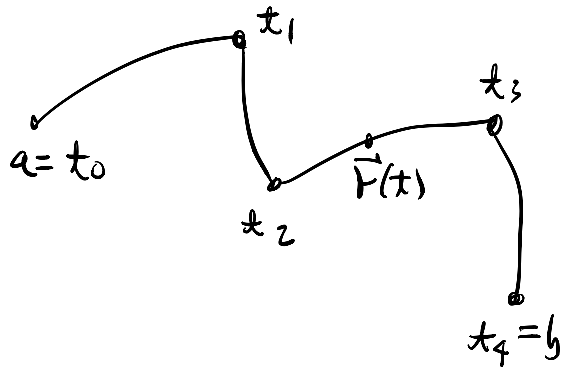
Then

$$ds = |\vec{r}'(t)|\,dt = \left|\frac{d\vec{r}}{dt}(t)\right| dt$$

$$= \left|\frac{d\vec{r}}{d\tilde{t}} \cdot \frac{d\tilde{t}}{dt}\right| dt = \left|\frac{d\vec{r}}{d\tilde{t}}\right| d\tilde{t}$$

$\therefore$ $ds$ and hence $\int_C f(\vec{r})\,ds$ is independent of the parametrization of $C$.

(3) If $\vec{r}(t)$ is only piecewise differentiable, then the RHS of Def 9' becomes a sum:



if $[a,b] = [t_0, t_1] \cup \cdots \cup [t_{i-1}, t_i] \cup \cdots \cup [t_{k-1}, t_k]$ (with $t_0 = a$, $t_k = b$)

( non-differentiable point for $\vec{r}(t)$ )

such that $\vec{r}\Big|_{[t_{i-1}, t_i]}$ is differentiable, then

$$\boxed{\int_C f(\vec{r})\,ds = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$$

<u>eg32</u>: $f(x,y,z) = x - 3y^2 + z$

$C$ = line segment joining the origin and $(1,1,1)$

Find $\int_C f(x,y,z)\,ds$

<u>Soln</u>: Parametrize $C$ by

$$\vec{r}(t) = t(1,1,1) = (t,t,t), \quad t \in [0,1]$$

(i.e. $x(t) = t$, $y(t) = t$, $z(t) = t$)

$$\Rightarrow \quad \vec{r}\,'(t) = (1,1,1), \quad \forall t \in [0,1]$$

$$\Rightarrow \quad |\vec{r}\,'(t)| = \sqrt{3}$$

Hence $\int_C f\,ds = \int_0^1 f(t,t,t)\sqrt{3}\,dt$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\,dt = 0 \quad \text{(check!)}$$

#

eg33: Let $C$ be a curve in $\mathbb{R}^2$ (plane curve) (i.e. $z(t)\equiv 0$) and it has 2 parametrizations

$$\vec{r}_1(t) = (\cos t, \sin t), \qquad t\in[-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\vec{r}_2(t) = (\sqrt{1-t^2}, -t), \qquad t\in[-1, 1]$$

Suppose $f(x,y)=x$. Find $\int_C f(x,y)\,ds$.

(We simply omit the $z$-variable, as $C$ is a plane curve and $f$ is indep. of $z$)

Soln: (i) $\vec{r}_1(t) = (\cos t, \sin t)$, $-\frac{\pi}{2}\le t\le \frac{\pi}{2}$

$$\int_C f(x,y)\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos t, \sin t)\left|(\cos t, \sin t)'\right| dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot \left|(-\sin t, \cos t)\right| dt$$

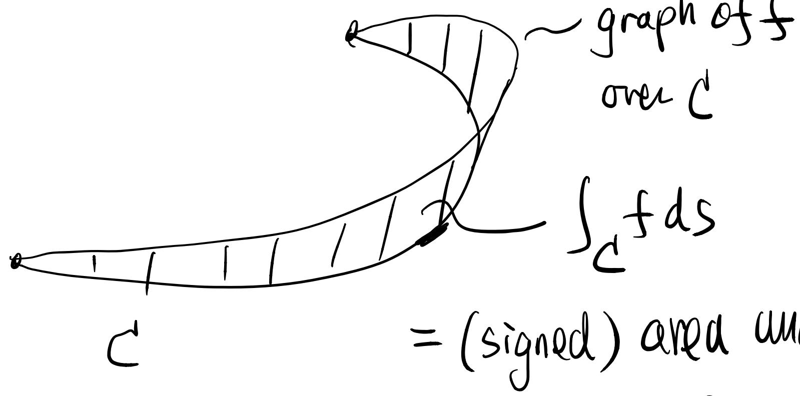
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\,dt = 2 \qquad \text{(check!)}$$

(ii) $\vec{r}_2(t) = (\sqrt{1-t^2}, -t)$, $-1\le t\le 1$

$$\int_C f(x,y)\,ds = \int_{-1}^{1} \sqrt{1-t^2}\sqrt{\left(\frac{d}{dt}\sqrt{1-t^2}\right)^2+\left[\frac{d}{dt}(-t)\right]^2}\,dt$$

$$= \cdots = \int_{-1}^{1} dt = 2 \qquad \text{(check!)}$$

This verifies the fact that the line integral is indep. of the parametrization




= (signed) area under the graph of f over C.

\#

**Prop 7**: If C is a piecewise smooth curve made by joining $C_1, C_2, \dots C_n$ end-to-end, then

$$\int_C f\,ds = \sum_{i=1}^{n} \int_{C_i} f\,ds$$

(Pf: Clear from the remark (3) of Def 9', but $C_i$ can be piecewise in this Prop.)

Remark: "end-to-end" means

"end point of $C_{k-1}$ = initial point of $C_k$".