Pf of Thm6 :

Thm 6 : Suppose $\phi : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ is a diffeomorphism (1-1, onto, s.t. ϕ and $\phi^{-1} \in C^1$) mapping a region G (closed and bounded) in the uv-plane onto a region R (closed and bounded) in the xy-plane (except possibly on the boundary). Suppose $f(x,y)$ is <u>continuous</u> on R, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\circ\phi(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

<u>Step 0</u> : We need better notations and terminology:

In this proof, we'll denote

$$D\phi = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$ the <u>differential</u> or <u>Jacobian matrix</u>

and $$\frac{\partial(x,y)}{\partial(u,v)} = \det D\phi$$ the <u>Jacobian determinant</u>

We also use "index" notations for the variables:

(x_1, x_2) or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ [instead of (x,y) or $\begin{pmatrix} x \\ y \end{pmatrix}$.]

Step 1 : Let $F : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$ near a point p

with $\dfrac{\partial(f_1, f_2)}{\partial(x_1, x_2)} \neq 0$ at p. Then, near the point p,

F can be decomposed into $F = H \circ K$

with H, K of the forms

$$K : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} k(x_1, x_2) \\ x_2 \end{pmatrix} \overset{\text{denote}}{=} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\left(\text{or } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} k(x_1, x_2) \\ x_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)$$

and $H : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ h(y_1, y_2) \end{pmatrix}$,

such that $\det DK \neq 0$ and $\det DH \neq 0$ at p.

Pf of Step 1 : By assumption $0 \neq \dfrac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$ at p

Case 1 $\dfrac{\partial f_1}{\partial x_1}(p) \neq 0$

Define $k(x_1, x_2) = f_1(x_1, x_2)$ near p.

Then the transformation

$$K : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 = f_1(x_1, x_2) \\ y_2 = x_2 \end{pmatrix}$$

is of the required form and has Jacobian matrix

$$DK = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \det DK(p) = \frac{\partial f_1}{\partial x_1}(p) \neq 0.$$

By <u>Inverse Function Theorem</u>, K is invertible near p and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = K^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} g(y_1, y_2) \\ x_2 \end{pmatrix}$$ is differentiable at $K(p)$ (since $x_2 = y_2$)

with $\quad D(K^{-1})_{K(p)} \cdot DK_p = Id \quad$ (chain rule)

i.e.

$$\underset{(\text{at } K(p))}{\begin{pmatrix} \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \\ 0 & 1 \end{pmatrix}} \underset{(\text{at } p)}{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow \quad \frac{\partial g}{\partial y_1} \cdot \frac{\partial f_1}{\partial x_1} = 1 \quad \text{and} \quad \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_2} + \frac{\partial g}{\partial y_2} = 0$$

In particular $\quad \det D(K^{-1})_{K(p)} = \dfrac{1}{\det DK_p} \neq 0.$

Now define $\quad h(y_1, y_2) = f_2(x_1, x_2) = f_2 \circ K^{-1}(y_1, y_2)$

$$= f_2(g(y_1, y_2), y_2)$$

and
$$H: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 = y_1 \\ z_2 = h(y_1, y_2) \end{pmatrix}$$

H is of the required form.

$$DH = \begin{pmatrix} \frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} \\ \frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial z_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} \end{pmatrix}$$

$$\det DH = \frac{\partial h}{\partial y_2} = \frac{\partial f_2}{\partial x_1}\frac{\partial x_1}{\partial y_2} + \frac{\partial f_2}{\partial x_2}\frac{\partial x_2}{\partial y_2}$$

$$= \frac{\partial f_2}{\partial x_1} \cdot \frac{\partial g}{\partial y_2} + \frac{\partial f_2}{\partial x_2} \cdot 1$$

$$= \frac{\partial f_2}{\partial x_1}\left(- \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_2}\right) + \frac{\partial f_2}{\partial x_2}$$

$$= - \frac{\frac{\partial f_2}{\partial x_1} \cdot \frac{\partial f_1}{\partial x_2}}{\frac{\partial f_1}{\partial x_1}} + \frac{\partial f_2}{\partial x_2}$$

$$= \frac{1}{\frac{\partial f_1}{\partial x_1}}\left[\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\right]$$

$$= \frac{\det DF}{\frac{\partial f_1}{\partial x_1}} \neq 0 \text{ at } p$$

So H and K satisfy the requirements and we have

$$H \circ K \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = H \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ h(y_1, y_2) \end{pmatrix}$$

$$= \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This completes the proof of case 1.

Case 2 $\frac{\partial f_1}{\partial x_1}(p) = 0$

Since $\det DF = \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} \neq 0$ at p,

$$\frac{\partial f_1}{\partial x_2}(p) \neq 0 .$$

Interchanging the variables $\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$

Then the new mapping $\tilde{F}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$ satisfies the condition in case 1. Applying case 1 to $\tilde{F}$, then interchanging back to x_1, x_2 ※

Step 2: Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k(x_1,x_2) \\ x_2 \end{pmatrix}$ be a diffeomorphism from region R_1 to $R_2 = K(R_1)$. Then for any function $f(y_1,y_2)$ on R_2,

$$\iint_{R_2} f(y_1,y_2)\,dy_1 dy_2 = \iint_{R_1} f\circ K(x_1,x_2)\,|\det DK|\,dx_1 dx_2$$

$$= \iint_{R_1} f\circ K(x_1,x_2)\left|\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}\right| dx_1 dx_2$$

Remark: Step 2 also holds for the other 2 types of transformations in Step 1. (Ex!)

Pf: By additivity property of integrations and cutting R_1 (and correspondingly $R_2 = K(R_1)$) into small regions, we may assume

$$R_1 = [a,b]\times[c,d] = \{a \le x_1 \le b,\ c \le x_2 \le d\}$$

For any fixed $y_2 = x_2$

$$y_1 = k(x_1, x_2) = k(x_1, y_2) \quad \text{for } a \le x_1 \le b$$

can be regarded as a transformation of 1-variable

Note that $\dfrac{\partial y_1}{\partial x_1} = \dfrac{\partial k}{\partial x_1} = \det\begin{pmatrix} \frac{\partial k}{\partial x_1} & \frac{\partial k}{\partial x_2} \\ 0 & 1 \end{pmatrix}$

$$= \det DK \neq 0$$

Note also that R_2 is of special form

$$\{c \le y_2 \le d,\ k(a,y_2) \le y_1 \le k(b,y_2)\} \qquad \left(\text{if } \frac{\partial y_1}{\partial x_1} > 0\right)$$

or $\{c \le y_2 \le d,\ k(b,y_2) \le y_1 \le k(a,y_2)\} \qquad \left(\text{if } \frac{\partial y_1}{\partial x_1} < 0\right)$

By Fubini's Thm (assuming $\frac{\partial y_1}{\partial x_1} > 0$, the other case is similar)

$$\iint_{R_2} f(y_1,y_2)\,dy_1 dy_2 = \int_c^d \left(\int_{k(a,y_2)}^{k(b,y_2)} f(y_1,y_2)\,dy_1 \right) dy_2$$

and change of variable famula in 1-variable implies

$$\int_{k(a,y_2)}^{k(b,y_2)} f(y_1,y_2)\,dy_1 = \int_a^b f(k(x_1,y_2),y_2)\,\frac{\partial y_1}{\partial x_1}\,dx_1$$

($\smile > 0$)

$$= \int_a^b f(k(x_1,x_2),x_2)\,\frac{\partial y_1}{\partial x_1}\,dx_1$$

(since $x_2 = y_2$ is fixed)

$\Rightarrow$

$$\iint_{R_2} f(y_1,y_2)\,dy_1\,dy_2 = \int_c^d \left(\int_a^b f(k(x_1,x_2),x_2)\,\frac{\partial y_1}{\partial x_1}\,dx_1 \right) dx_2$$

$$= \int_c^d \int_a^b f(k(x_1,x_2),x_2)\,\left|\det DK\right|\,dx_1\,dx_2$$

(since $\frac{\partial y_1}{\partial x_1} > 0$)

$$= \iint_{R_1} f(k(x_1,x_2),x_2)\,\left|\det DK\right|\,dx_1\,dx_2$$

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$\boxed{\text{Step3 : If the change of variables famula holds for } F \,\&\, G\text{, then it holds for } F \circ G}$

Pf : Easily by $D(F\circ G) = DF \cdot DG$ (Chain Rule)

$$\Rightarrow \quad \left|\det D(F\circ G)\right| = \left|\det DF\right|\,\left|\det DG\right|.$$

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(Details leave as Ex!)

Final Step: Combining steps 1–3, and using additivity property of integration, we've proved the Thm 6 for general change of variable formula ✗

(need MATH3060)

[Actually, this applies to all dimensions.]