eg31 Let $D = \{(x,y,z) \in \mathbb{R}^3 : |x|+|y|+|z| \le 1\}$

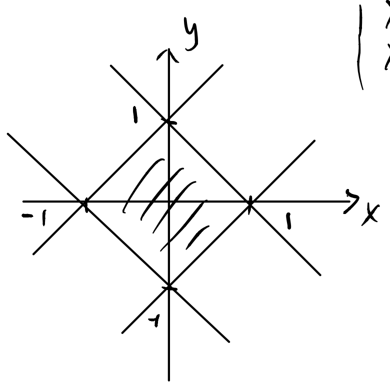
Evaluate $\iiint_D (x+y+z)^4 \, dV$.

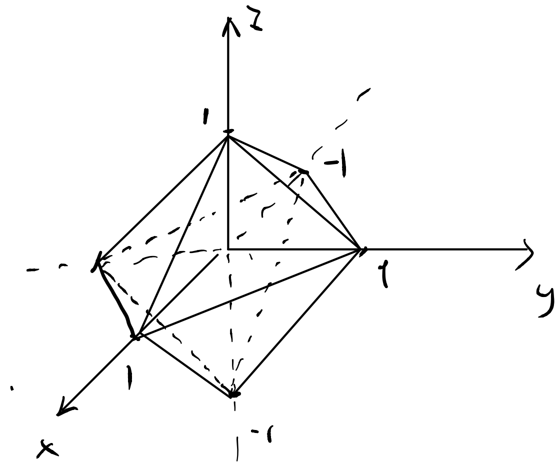
(can use symmetric $(x,y,z) \leftrightarrow (-x,-y,-z)$
to reduce half, but not to the 1st octant
since for instance $x+y+z \leftrightarrow x+y-z$
under $(x,y,z) \leftrightarrow (x,y,-z)$

$(x+y+z)^4$ is not symmetric in all reflection with respect to the coordinates lines)

Soln: If $z=0$,
then $|x|+|y| \le 1$

$$\begin{cases} x+y=\pm 1 \\ x-y=\pm 1 \end{cases}$$





Boundary surfaces are given by
$\pm x \pm y \pm z = 1$ (8 surfaces)

Let
$$\begin{cases} u = x+y+z \\ v = x+y-z \\ w = x-y-z \end{cases}$$

boundary planes
$$\begin{cases} u = \pm 1 \\ v = \pm 1 \\ w = \pm 1 \end{cases}$$
only 6 out of 8 surfaces.

need to find formula for other 2 boundary surfaces.

Boundary planes

$\pm x \pm y \pm z = 1$ $\quad \begin{pmatrix} + & + & + \\ - & - & - \end{pmatrix}$ $\quad u = \pm 1$

$\begin{pmatrix} + & + & - \\ - & - & + \end{pmatrix} \longleftrightarrow v = \pm 1$

$\begin{pmatrix} + & - & - \\ - & + & + \end{pmatrix}$ $\quad w = \pm 1$

remaining pair of bdy planes $\begin{pmatrix} + & - & + \\ - & + & - \end{pmatrix}$ $\quad \boxed{u - v + w = \pm 1 \text{ (check!)}}$

Change of variable famula $\Rightarrow$

$$\iiint_D (x+y+z)^4 \, dV = \iiint_{\substack{-1 \le u, v, w \le 1 \\ -1 \le u - v + w \le 1}} u^4 \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dv \, dw \, du$$

By solving $\begin{cases} u = x + y + z \\ v = x + y - z \\ w = x - y - z \end{cases}$

we have $\begin{cases} x = \frac{1}{2}(u+w) \\ y = \frac{1}{2}(v-w) \\ z = \frac{1}{2}(u-v) \end{cases}$ (check!)

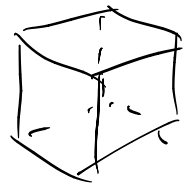
$$\Rightarrow \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\det\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}\right| = \left|-\frac{1}{4}\right| = \frac{1}{4} \quad \text{(check!)}$$

Hence

$$\iiint_D (x+y+z)^4\, dV = \iiint_{\substack{-1 \le u,v,w \le 1 \\ -1 \le u-v+w \le 1}} \frac{u^4}{4}\, dv\, dw\, du$$

$$= A - B - C$$

where $$A = \iiint_{-1 \le u,v,w \le 1} \frac{u^4}{4}\, dv\, dw\, du$$



$\{-1 \le u,v,w \le 1\}$

$$B = \iiint_{\substack{-1 \le u,v,w \le 1 \\ u-v+w \ge 1}} \frac{u^4}{4}\, dv\, dw\, du$$

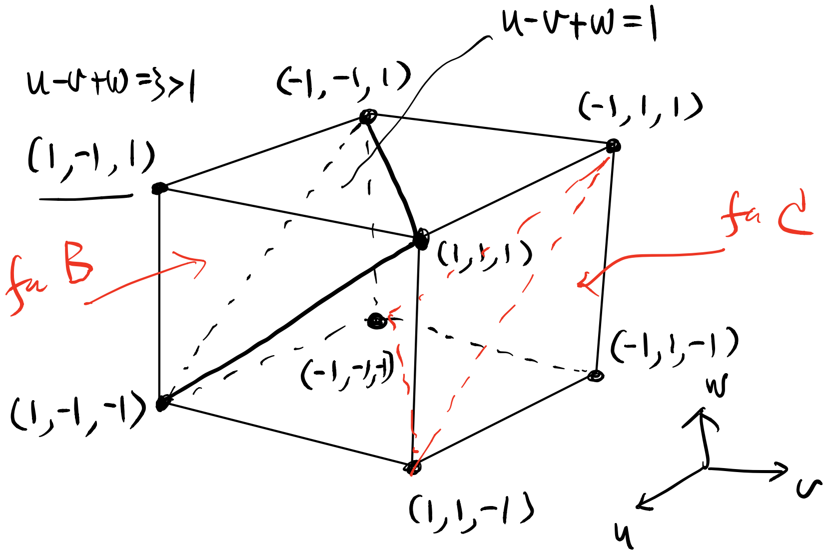
$$C = \iiint_{\substack{-1 \le u,v,w \le 1 \\ u-v+w \le -1}} \frac{u^4}{4}\, dv\, dw\, du$$

(observation: $B = C$ by symmetric $(u,v,w) \Leftrightarrow (-u,-v,-w)$)

It is clear that

$$A = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{u^4}{4}\, dv\, dw\, du \overset{\text{easy}}{=\!=} \frac{2}{5} \text{ (check!)}$$

To handle B & C



	$u-v+w$
$(1,-1,1)$	$1-(-1)+1=3$
$(1,1,1)$	$1-1+1=1$
$(-1,-1,1)$	$(-1)-(-1)+1=1$
$(1,-1,-1)$	$1-(-1)+(-1)=1$

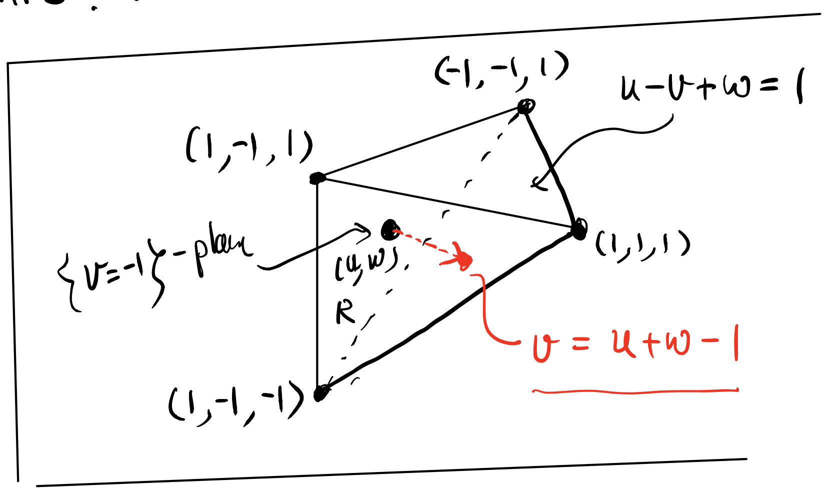
Hence the 3 points $(1,1,1)$, $(-1,-1,1)$, $(1,-1,-1)$ are on the boundary plane $\boxed{u-v+w=1}$

Since plane determined by 3 points, so $\{u-v+w=1\}$ is the plane passing thro. these 3 vertexes.

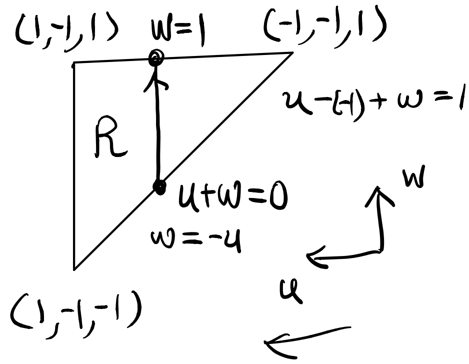
So the solid region

$$\left\{\begin{array}{l} -1 \le u, v, w \le 1 \\ u-v+w \ge 1 \end{array}\right\}$$

for integration B is as figure, which is of special type



$$\therefore \quad B = \iint_R \left[\int_{-1}^{u+w-1} \frac{u^4}{4}\,dv \right] dw\,du$$



$$= \int_{-1}^{1} \left(\int_{-u}^{1} \left[\int_{-1}^{u+w-1} \frac{u^4}{4}\,dv \right] dw \right) du$$

$$= \cdots = \frac{3}{35} \quad \text{(check!)}$$

Then symmetric, the solid for the integration C is determined by the other 4 vertexes

$(-1,1,-1)$, $(-1,-1,-1)$, $(1,1,-1)$, & $(-1,1,1)$

and

$$C = B = \frac{3}{35} \quad \text{also.}$$

Finally

$$\iiint_D (x+y+z)^4\,dV = A - B - C$$

$$= \frac{2}{5} - \frac{3}{35} - \frac{3}{35} = \frac{8}{35}$$

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