Application of Multiple Integrals (Thomas' Calculus §15.6)

In applications, we often use the following:

In 2-dim: Let R be a region in $\mathbb{R}^2$ with <u>density</u> $\delta(x,y)$

- First moment about y-axis: $M_y = \iint_R x\,\delta(x,y)\,dA$
- First moment about x-axis: $M_x = \iint_R y\,\delta(x,y)\,dA$
- Mass: $M = \iint_R \delta(x,y)\,dA$
- Center of Mass (Centroid)

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

In 3-dim, D solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

- <u>First moment</u>:
  - about yz-plane, $M_{yz} = \iiint_D x\,\delta(x,y,z)\,dV$
  - about xz-plane, $M_{xz} = \iiint_D y\,\delta(x,y,z)\,dV$
  - about xy-plane, $M_{xy} = \iiint_D z\,\delta(x,y,z)\,dV$
- Mass: $M = \iiint_D \delta(x,y,z)\,dV$
- <u>Center of Mass</u> (Centroid) $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$

In 2-dim, R = region in $\mathbb{R}^2$ with density $\delta(x,y)$

Moments of inertia

- about x-axis : $I_x = \iint_R y^2 \delta(x,y)\, dA$
- about y-axis : $I_y = \iint_R x^2 \delta(x,y)\, dA$
- about line L : $I_L = \iint_R r(x,y)^2 \delta(x,y)\, dA$

  where $r(x,y)$ = distance between (x,y) and L.

- about the origin : $I_0 = \iint_R (x^2+y^2)\delta(x,y)\, dA$

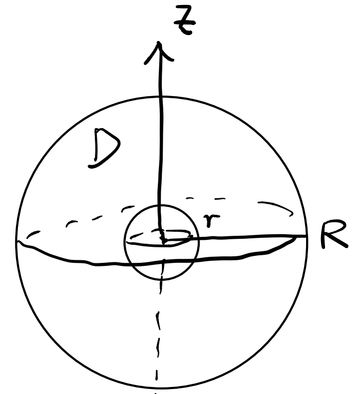
In 3-dim, D = solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

Moments of Inertia

- around x-axis : $I_x = \iiint_D (y^2+z^2)\,\delta(x,y,z)\, dV$
- around y-axis : $I_y = \iiint_D (x^2+z^2)\,\delta(x,y,z)\, dV$
- around z-axis : $I_z = \iiint_D (x^2+y^2)\,\delta(x,y,z)\, dV$
- around Line L : $I_L = \iiint_D r(x,y,z)^2\,\delta(x,y,z)\, dV$

  where $r(x,y,z)$ = distance between (x,y,z) and L.

eg 27: Consider $D: \; r^2 \le x^2+y^2+z^2 \le R^2$
$(0 < r < R)$



with density $\delta(x,y,z) \equiv \delta$

(constant density function, i.e. uniform mass)

Express I_z in term of $m =$ Mass of D, r and R.

Soln: $$I_z \overset{\text{def}}{=} \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV$$

$$= \delta \iiint_D (x^2+y^2)\,dV$$

$$= \delta \int_0^{2\pi}\int_0^{\pi}\int_r^{R} (\rho\sin\phi)^2 \cdot \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \delta\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin^3\phi\, d\phi\right)\left(\int_r^R \rho^4\, d\rho\right)$$

$$= \frac{8\pi}{15}(R^5-r^5)\,\delta$$

Mass $$m = \iiint_D \delta(x,y,z)\,dV = \delta\iiint_D dV$$

$$= \delta\,\frac{4\pi}{3}(R^3-r^3) \qquad \text{(check!)}$$

$$\Rightarrow \boxed{I_z = \frac{2m}{5}\,\frac{R^5-r^5}{R^3-r^3}}.$$

<u>Observation</u>: Two limiting cases:

(i) $r \to 0$, i.e. the whole solid ball $\boxed{I_z = \frac{2m}{5} R^2.}$

(ii) $r \to R$, i.e. a (hollow) sphere made of "infinitesimally" thin sheet:

$$I_z = \lim_{r \to R} \frac{2m}{5} \cdot \frac{R^5 - r^5}{R^3 - r^3} = \frac{2m}{5} \cdot \frac{5R^4}{3R^2} \quad \text{(check!)}$$

$$\therefore \quad \boxed{I_z = \frac{2m}{3} R^2}$$

Moment of inertia of the <u>hollow sphere</u>

> moment of inertia of the <u>solid ball</u>

(assuming the same (uniform) mass m) ##

Change of Variable Formula

Review of 1-variable

$$\left(\begin{aligned} &\int_a^b F'(x)\,dx = F(b)-F(a) \\ &\int_a^b f(x)\,dx = \text{limit of Riemann sum.} \end{aligned} \right)$$

In Riemann sum

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,\underbrace{dx}_{}\ \left(\sim |\Delta x| = \text{length of subinterval} > 0\right)$$

$[a,b]$ as set (we don't care about the direction)

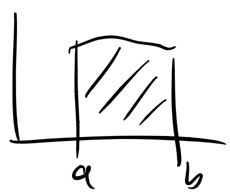
If $a>b$

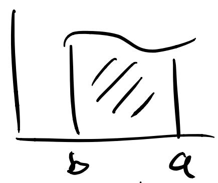
same set

$$\int_a^b f(x)dx = -\int_b^a f(x)dx = -\int_{\underbrace{[b,a]}} f(x)\,dx$$

In summary

$$\int_a^b f(x)dx = \begin{cases} \displaystyle\int_{[a,b]} f(x)dx, & \text{if } a \le b \\ \displaystyle -\int_{[a,b]} f(x)dx, & \text{if } a \ge b \end{cases}$$





($[a,b]$ $\|$ $[b,a]$ as set : $\{x : x \text{ between } a \,\&\, b\}$)

Change of variable in 1-variable

$$\int_a^b f(x)dx = \int_c^d \left[f(x(u))\frac{dx}{du}\right] du$$

where $c = u(a)$, $d = u(b)$.

If $\frac{dx}{du} > 0$, then $d = u(b) > u(a) = c$

$$\therefore \int_a^b f(x)\,dx = \int_{[c,d]} \left[f(x(u)) \frac{dx}{du} \right] du$$

$$= \int_{[c,d]} f(x(u)) \left| \frac{dx}{du} \right| du$$

If $\frac{dx}{du} < 0$, then $d = u(b) < u(a) = c$

$$\therefore \int_a^b f(x)\,dx = \int_c^d \left[f(x(u)) \frac{dx}{du} \right] du$$

$$= - \int_{[d,c]} f(x(u)) \frac{dx}{du}\, du$$

$$= \int_{[d,c]} f(x(u)) \left| \frac{dx}{du} \right| du$$

Hence (in Riemann sum)

$\frac{|\Delta x|}{|\Delta u|} \sim \left| \frac{dx}{du} \right|$

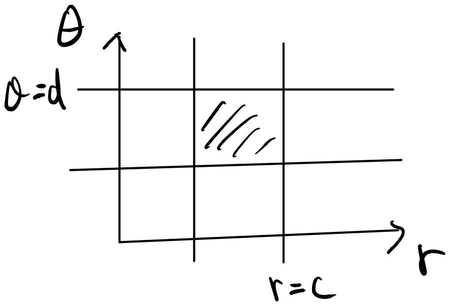
$$\boxed{\int_{[a,b]} f(x)\,dx = \int_{[c,d]} f(x) \left| \frac{dx}{du} \right| du}$$

interpreted as a set without direction

(i.e. $\{x : x$ between c & d (inclusive)$\}$)

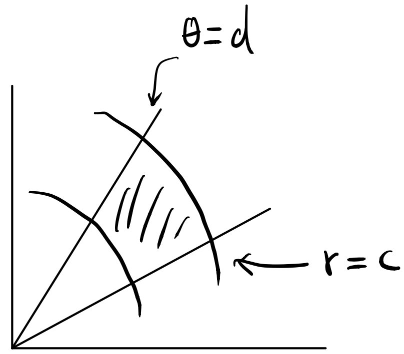
Back to multiple integrals

Recall: Polar coordinates

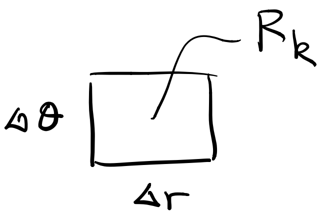


$\xrightarrow{\phi}$

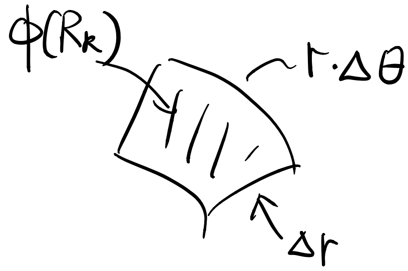
$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$



$\phi(r,\theta) = (x,y)$



$\xrightarrow{\phi}$



$\text{Area}(R_k) \cong \Delta r \Delta\theta$

$\text{Area}(\phi(R_k)) \cong r \Delta r \Delta\theta$

$$\frac{\text{Area}(\phi(R_k))}{\text{Area}(R_k)} \to r \quad \text{as "} R_k \to \text{point"}$$

ratio of area, always > 0