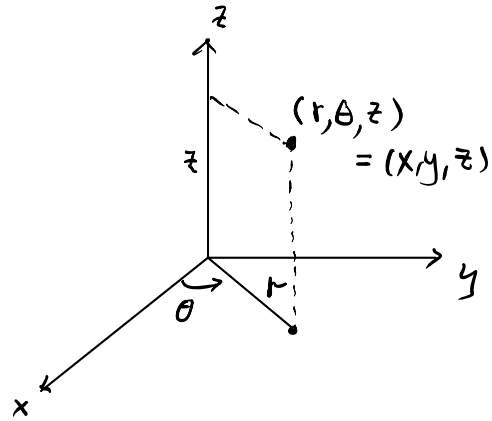
Cylindrical Coordinates in $\mathbb{R}^3$



- (r,θ) = polar coordinates for the xy-plane
  $(r \geq 0)$

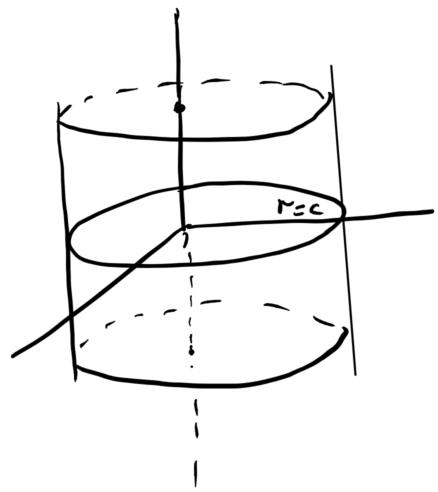
- z = rectangular vertical coordinate

Then a point $P = (x,y,z)$ can be represented by (r,θ,z), where

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

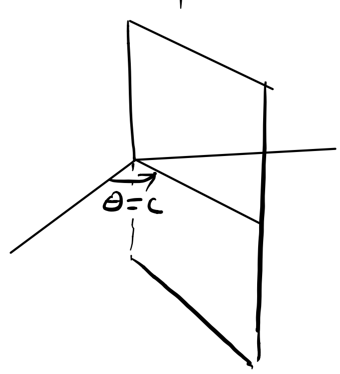
And (r,θ,z) is called the <u>cylindrical coordinates</u> for $\mathbb{R}^3$.

<u>Remark 1</u>: (Let c be a constant)



- $r = c$ $(c > 0)$ describes a <u>cylinder</u>

- $\theta = c$ $(0 \leq c \leq 2\pi)$ describes a vertical half-plane



- $z = c$ describes a horizontal plane (as in rectangular coordinates)

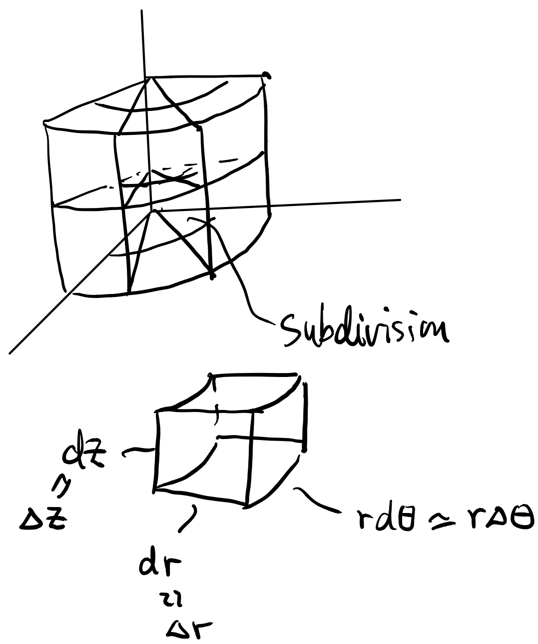
Remark 2 : We can define cylindrical coordinates in other directions.

e.g. $\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases}$ (H.W. draw the cylinder $r = c$)

Volume element

$$dV = \underbrace{dx\,dy}_{\updownarrow}\,\underbrace{dz}_{\updownarrow}$$

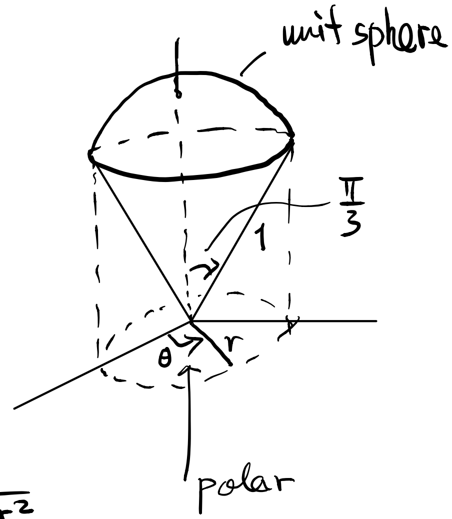
$$= r\,dr\,d\theta \cdot dz$$



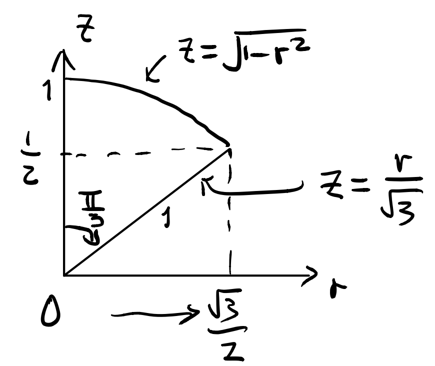
(order of the integration can be changed)

eg22 (see also eg24)

Find the volume of the Ice-cream cone I given as in the figure.



Soln: θ fixed



$$\left(\begin{aligned} z &= \sqrt{1-r^2} \\ &= f(r,\theta) \end{aligned} \right)$$
↑ indep. of θ

Fubini's $\Rightarrow$ $\mathrm{Vol}(D) = \displaystyle\int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} r\,dz\,dr \cdot d\theta$

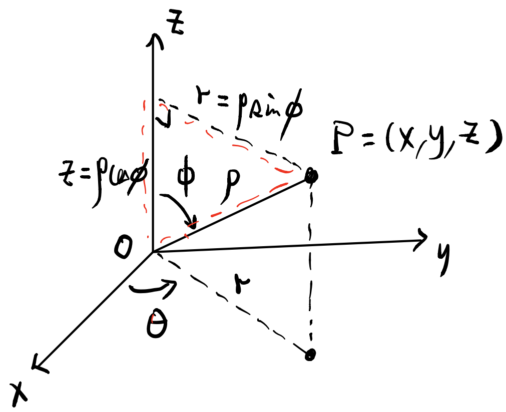
don't miss this factor

(1st z then r)

$$= 2\pi \underbrace{\int_0^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{\sqrt{3}}\right) r\,dr}_{\text{indep. of } \theta}$$

$$= \cdots = \frac{\pi}{3} \quad \text{(check!)}$$

Spherical coordinates in $\mathbb{R}^3$



(ρ, ϕ, θ) where

- ρ = distance from the origin $(\rho \geq 0)$

  (ρ = "rho")

- ϕ = angle from the <u>positive</u> z-axis to $\overline{OP}$ $(0 \leq \phi \leq \pi)$
- θ = angle from cylindrical coordinate $(0 \leq \theta \leq 2\pi)$

<u>Remark</u>: If (r, θ, z) is the cylindrical coordinates of the point P, then

$$\begin{cases} r = \rho\sin\phi \\ z = \rho\cos\phi \end{cases}$$

In particular $z^2 + r^2 = \rho^2$.

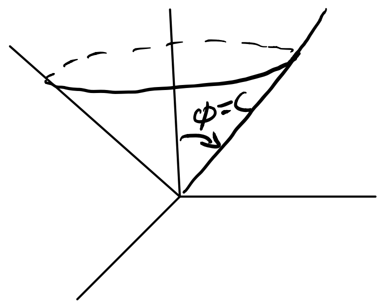
Then

$$\boxed{\begin{array}{ccccc} x & = & r\cos\theta & = & \rho\sin\phi\cos\theta \\ y & = & r\sin\theta & = & \rho\sin\phi\sin\theta \\ z & = & z & = & \rho\cos\phi \end{array}}$$

rectangular cylindrical spherical

Remark: If c is a constant, then

- $\rho = c$ $(c>0)$ describes a sphere of radius c
- $\theta = c$ describes a vertical half-plane.
- $\phi = c$ describes



$$= \begin{cases} +\text{ve } z\text{-axis}, & \text{if } c=0 \\ -\text{ve } z\text{-axis}, & \text{if } c=\pi \\ xy\text{-plane}, & \text{if } c=\frac{\pi}{2} \\ \text{cone}, & \text{otherwise} \end{cases}$$

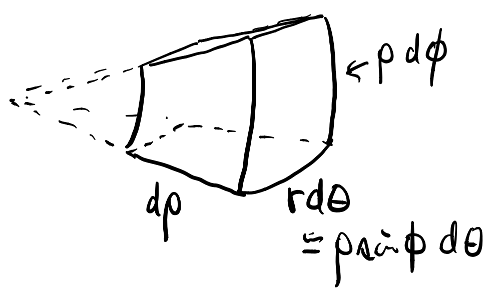
$\left(\begin{array}{l} \text{upward } 0<c<\frac{\pi}{2} \\ \text{downward } \frac{\pi}{2}<c<\pi \end{array}\right)$

<u>Volume element</u>

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$

$$= (\rho\sin\phi)(\rho\,d\rho\,d\phi)\,d\theta$$

i.e. $\boxed{dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$



<u>eg 23</u> Convert the following into spherical coordinates

(1) $x^2+y^2+(z-1)^2=1$ (sphere)

(2) $z = -\sqrt{x^2+y^2}$ (cone)

Soln: (1) Sub. $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$

into $x^2+y^2+(z-1)^2 = 1$

$\Rightarrow \quad \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 = 1$

$\Leftrightarrow \quad \rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi = 0$

$\Leftrightarrow \quad \rho^2 = 2\rho\cos\phi$

$\Leftrightarrow \quad \rho = 2\cos\phi$ (since $\rho \geq 0$ & $\rho = 0$ is a point.)

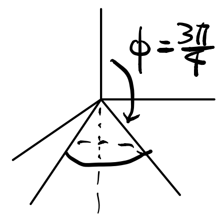
(2) Sub. the formula into $z = -\sqrt{x^2+y^2}$ $(=-r)$

$\Rightarrow \quad \rho\cos\phi = -\rho\sin\phi$ $\left(\begin{array}{l} \rho \geq 0 \\ 0 \leq \phi \leq \pi \Rightarrow \sin\phi \geq 0 \end{array}\right)$

For $\rho \neq 0$ (i.e. not the origin)

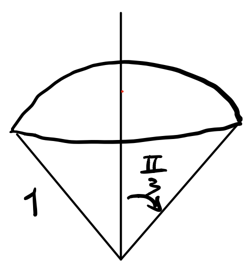
$\cos\phi = -\sin\phi$

$\Rightarrow \quad \phi = \frac{3\pi}{4}$ (in the range)



eg24 (see eg22)

Volume of ice-cream cone I again, in spherical coordinates



Soln: The ice-cream cone I is given by

$\{0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \frac{\pi}{3}, \quad 0 \leq \theta \leq 2\pi\}$

$$\Rightarrow \operatorname{vol}(I) = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \underbrace{\sin\phi}_{} \, d\rho \, d\phi \, d\theta$$

don't miss this.

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin\phi \, d\phi\right)\left(\int_0^1 \rho^2 d\rho\right)$$

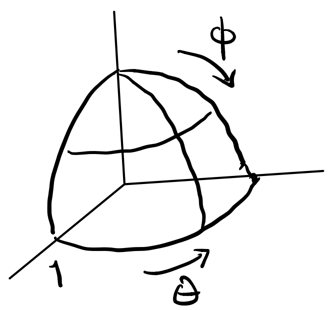
$$= \frac{\pi}{3} \quad (\text{Check!}) \quad ※$$

eg25

$$f(x,y,z) = \begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} &, \text{ if } (x,y,z) \neq (0,0,0) \\ 0 &, \text{ if } (x,y,z) = (0,0,0) \end{cases}$$

(In fact, f is continuous, but it is sufficient to know f is continuous except the origin $(0,0,0)$)

Let D = unit ball centered at origin intersecting with the 1st octant



Then D can be represented in spherical coordinates:

$$\begin{cases} 0 \le \rho \le 1 \\ 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

And for $(x,y,z) \neq (0,0,0)$

$$f(x,y,z) = \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} = \frac{(\rho\sin\phi)^2}{\rho} = \rho\sin^2\phi$$

(as $\rho \to 0$, $f \to 0 \Rightarrow f$ is continuous at $(0,0,0)$)

Hence $\iiint_D f(x,y,z)\,dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \underbrace{(\rho \sin^2\phi)}_{f} \cdot \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{\text{volume element}}$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{\pi}{2} \left(\int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi \right) \left(\int_0^1 \rho^3 d\rho \right)$$

$$= \frac{\pi}{12} \quad \text{(check!)}$$

If we want to calculate the average of f over D, we need to calculate $Vol(D)$ too.

In our case $Vol(D) = \frac{1}{8} Vol(\text{unit sphere}) = \frac{1}{8} \cdot \frac{4\pi}{3} = \frac{\pi}{6}$

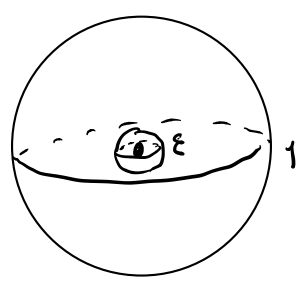
Hence average of f over D $= \frac{1}{Vol(D)} \iiint_D f(x,y,z)\,dV = \frac{1}{2}$

#

eg 26: (Improper integrals)

Let $f(x,y,z) = \frac{1}{x^2+y^2+z^2} = \frac{1}{\rho^2}$ (unbounded as $\rho \to 0$)

$g(x,y,z) = \frac{1}{(\sqrt{x^2+y^2+z^2})^3} = \frac{1}{\rho^3}$



over unit ball $B = \{(\rho,\theta,\phi) : 0 \le \rho \le 1\}$

(i) Does $\lim_{\varepsilon \to 0} \iiint_{B \setminus B_\varepsilon} f(x,y,z)\,dV$ exist?

where $B_\varepsilon = \{(\rho,\phi,\theta): 0 \le \rho \le \varepsilon\}$

(ii) Does $\displaystyle\lim_{\varepsilon\to 0} \iiint_{B\setminus B_\varepsilon} g(x,y,z)\,dV$ exist?

Answer: For $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2} = \dfrac{1}{\rho^2}$

$$\lim_{\varepsilon\to 0} \iiint_{B\setminus B_\varepsilon} f(x,y,z)\,dV = \lim_{\varepsilon\to 0} \int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^1 \frac{1}{\rho^2}\cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

(since $B\setminus B_\varepsilon = \{(\rho,\theta,\phi): \varepsilon < \rho \le 1\}$)

$$= \lim_{\varepsilon\to 0}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\, d\phi\right)\left(\int_\varepsilon^1 d\rho\right)$$

$$= \lim_{\varepsilon\to 0} 4\pi(1-\varepsilon) = 4\pi \quad \text{exists!}$$

For $g(x,y,z) = \dfrac{1}{(\sqrt{x^2+y^2+z^2})^3} = \dfrac{1}{\rho^3}$

$$\lim_{\varepsilon\to 0} \iiint_{B\setminus B_\varepsilon} g(x,y,z)\,dV = \lim_{\varepsilon\to 0} \int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^1 \frac{1}{\rho^3}\cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \lim_{\varepsilon\to 0}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\, d\phi\right)\left(\int_\varepsilon^1 \frac{1}{\rho}\, d\rho\right)$$

$$= \lim_{\varepsilon\to 0} 4\pi \ln\frac{1}{\varepsilon} \quad \text{doesn't exist!}$$

Terminology:

- $f = \dfrac{1}{\rho^2}$ is said to be "integrable" over B (in the sense of improper integral)

- $g = \frac{1}{\rho^3}$ is said to be "non integrable" over B

Question: determine all $\beta > 0$ such that

$f = \frac{1}{\rho^\beta}$ is "integrable" over $B \subset \mathbb{R}^3$

Similar question in $\mathbb{R}^2$: determine all $\beta > 0$ such that

$f = \frac{1}{r^\beta}$ is "integrable" in $\{r \le 1\} \subset \mathbb{R}^2$

(even in $\mathbb{R}^1$: $f = \frac{1}{|x|^\beta}$)