Remark: For D of type 1,

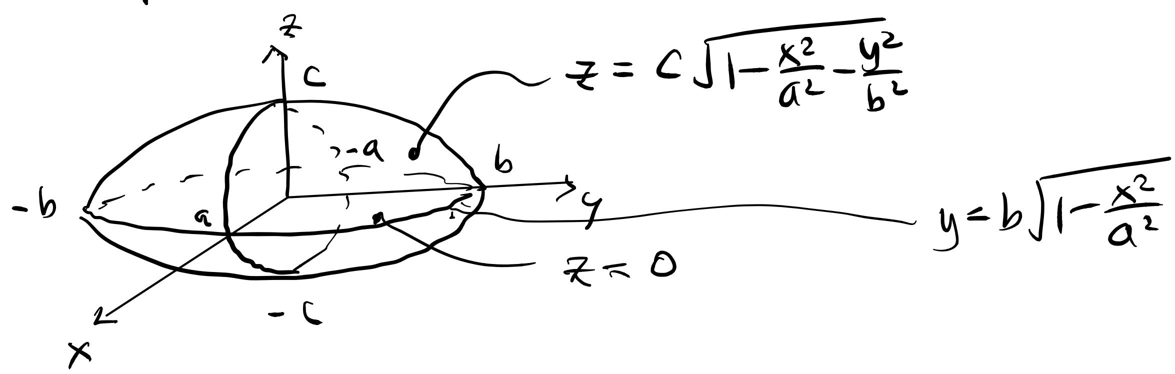
$$\mathrm{Vol}(D) = \iiint_D 1\, dV \overset{\text{Fubini}}{=} \iint_{R_1} \left[\int_{u_1(x,y)}^{u_2(x,y)} 1\, dz \right] dA$$

$$= \iint_{R_1} [u_2(x,y) - u_1(x,y)]\, dA$$

Formula for volume between two graphs $z = u_2(x,y)$ and $z = u_1(x,y)$.

eg18: Volume of Ellipsoid

$$D = \left\{ (x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\} \quad (a,b,c > 0)$$



By symmetry, we can consider the 1st octant only

$\Rightarrow$ $\mathrm{Vol}(D) = 8 \cdot$ volume of D in the 1st octant.

$$= 8 \cdot \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} 1\, dz \right] dy\, dx$$

$$= 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\, dx$$

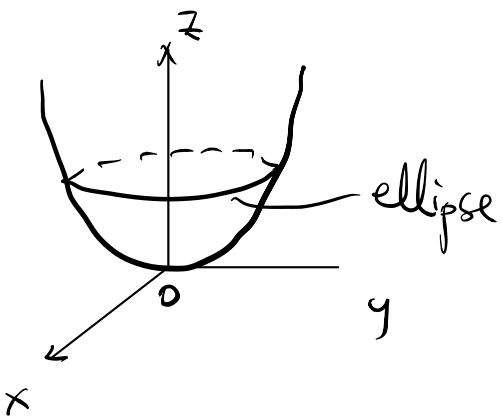
(volume between two graphs)

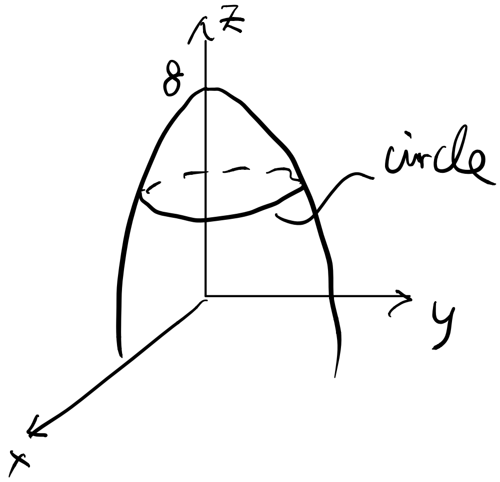
$= \cdots = \dfrac{4\pi abc}{3}$ (optional exercise)

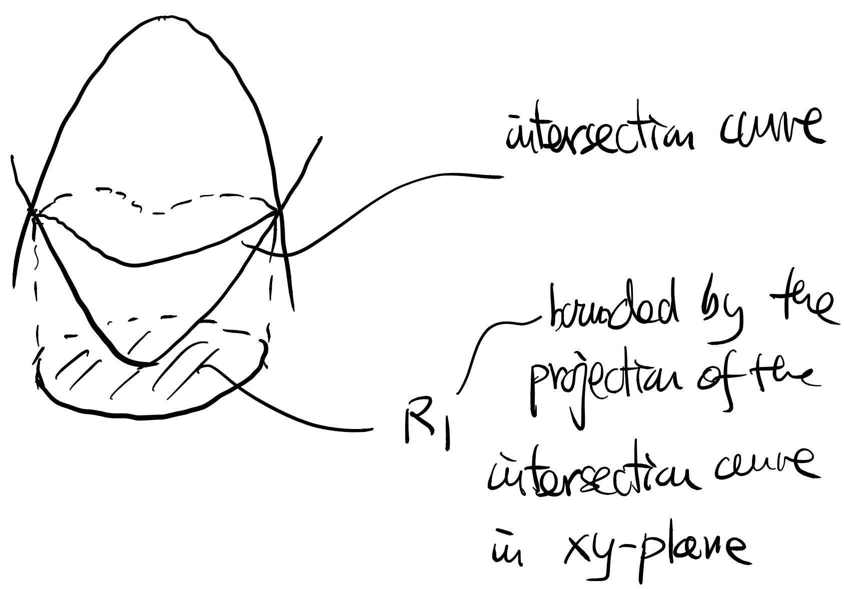
[In fact, we will have a better way to calculate this volume by "change of variables formula" (later)]

<u>eg19</u>: Find the volume of D enclosed by

$z = x^2+3y^2$ and $z = 8-x^2-y^2$







At the intersection of the two surfaces

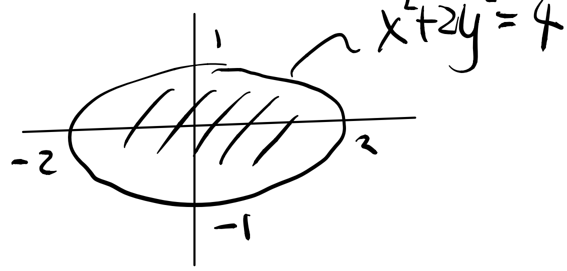
$$x^2+3y^2 = z = 8-x^2-y^2$$

$\Rightarrow$ $x^2+3y^2=8-x^2-y^2$ is the projection (in xy-plane) of the intersection curve

$\Rightarrow$ $x^2+2y^2=4$

(a ellipse)

So R_1 is



$$\Rightarrow \quad D=\left\{ (x,y)\in R_1 = \{x^2+2y^2\le 4\},\ x^2+3y^2\le z\le 8-x^2-y^2 \right\}$$

$$= \left\{ (x,y,z):\ -2\le x\le 2,\ -\sqrt{\frac{4-x^2}{2}}\le y\le \sqrt{\frac{4-x^2}{2}},\ x^2+3y^2\le z\le 8-x^2-y^2 \right\}$$

Fubini $\Rightarrow$

$$\mathrm{vol}(D)=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} 1\cdot dz\,dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \left[(8-x^2-y^2) - (x^2+3y^2) \right] dy\,dx$$

(double integral for volume between two graphs)

$$= \int_{-2}^{2} \frac{4\sqrt{2}}{3} (4-x^2)^{3/2} dx \quad \text{(check!)}$$

$$= 8\pi\sqrt{2} \quad \text{(check!)}.$$

[For those interested in the intersection (space) curve (in parametric form)

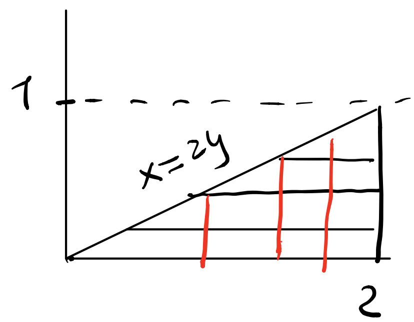
$x = 2\cos t$, $y = \sqrt{2}\sin t$, $z = 4 + 2\sin^2 t$
$(0 \le t \le 2\pi)$]

eg20 Evaluate $\displaystyle\int_0^4 \int_0^1 \int_{2y}^{2} \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\,dy\,dz$

$$= \int_0^4 \frac{2}{\sqrt{z}} \left[\int_0^1 \int_{2y}^{2} \cos(x^2)\, dx\,dy \right] dz$$

$$= \left[\int_0^1 \int_{2y}^2 \cos(x^2)\,dx\,dy\right] \cdot \left(\int_0^4 \frac{2}{\sqrt{z}}\,dz\right)$$

think of this as double integral over the region



By Fubini's

$$= \left[\int_0^2 \int_0^{\frac{x}{2}} \cos(x^2)\,dy\,dx\right] \cdot \left(\int_0^4 \frac{2}{\sqrt{z}}\,dz\right)$$

$$= \left[\int_0^2 \cos(x^2)\left(\int_0^{\frac{x}{2}} dy\right)dx\right] \cdot \int_0^4 \frac{2}{\sqrt{z}}\,dz$$

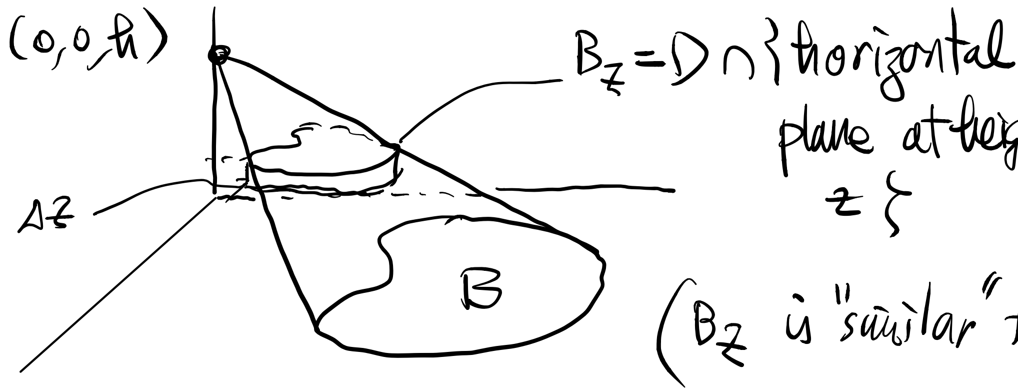
$$= \left(\int_0^2 \frac{x}{2}\cos(x^2)\,dx\right) \cdot \int_0^4 \frac{2}{\sqrt{z}}\,dz$$

Then the integration can be easily evaluated:

$= 2\sin 4$ (check!) #

eg 21 Let B (base) be a "nice" subset of $\mathbb{R}^2$.

Let D = cone in $\mathbb{R}^3$ with base B on xy-plane and vertex $(0,0,h)$ $(h>0)$



(B_z is "similar" to B)

How to find the volume of D?

Answer: by concept of Riemann sum and for this figure

$\text{Vol} \sim \text{Area}(B_z)\,\Delta z$

$$\Rightarrow \quad \text{Vol}(D) = \int_0^h \text{Area}(B_z)\,dz$$

By similarity: ratio of heights: $\dfrac{h-z}{h} = 1 - \dfrac{z}{h}$

$$\Rightarrow \text{ ratio of areas: } \frac{\text{Area}(B_z)}{\text{Area}(B)} = \left(1 - \frac{z}{h}\right)^2$$

$$\Rightarrow \quad \mathrm{Vol}(D) = \int_0^h \left(1-\frac{z}{h}\right)^2 \mathrm{Area}(B)\, dz$$

$$= \mathrm{Area}(B) \int_0^h \left(1-\frac{z}{h}\right)^2 dz$$

$$= \frac{h}{3}\,\mathrm{Area}(B) \qquad \text{(Check!)}$$

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