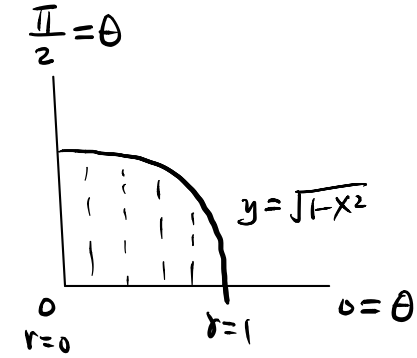
eg 13 Convert integrals between Cartesian and Polar coordinates

(a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta \cos\theta \, dr \, d\theta$

(b) $\int_1^2 \int_0^{\sqrt{2x-x^2}} y \, dy \, dx$



Soln: (a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta \cos\theta \, dr \, d\theta$

$= \int_0^{\frac{\pi}{2}} \left[\int_0^1 r^2 \sin\theta \cos\theta \, r dr \right] d\theta$

Region : $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$

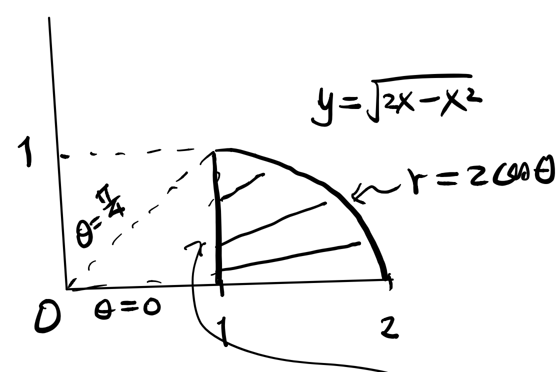
$\Rightarrow \int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta \cos\theta \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 (r^2 \sin\theta \cos\theta) \underbrace{r dr d\theta}$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx$

$\left(\text{or} = \int_0^1 \int_0^{\sqrt{1-y^2}} xy \, dx \, dy \right)$

(b) $\int_1^2 \int_0^{\sqrt{2x-x^2}} y \, dy \, dx$

$= \int_1^2 \left[\int_0^{\sqrt{2x-x^2}} y dy \right] dx$



Region is $1 \le x \le 2$, $0 \le y \le \sqrt{2x-x^2}$

$(0 \le \theta \le \frac{\pi}{4})$

The curve $x = 1$

$\Leftrightarrow r\cos\theta = 1 \Leftrightarrow r = \sec\theta$

The curve $y=\sqrt{2x-x^2}$

$\Leftrightarrow \quad r\sin\theta = \sqrt{2r\cos\theta - r^2\cos^2\theta}$

$\vdots$

$\Leftrightarrow \quad r^2 = 2r\cos\theta \qquad$ (check!)

$\Leftrightarrow \quad r = 2\cos\theta$

Hence $\displaystyle\int_1^2 \int_0^{\sqrt{2x-x^2}} y\,dy\,dx$

$$= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} (r\sin\theta)\, r\,dr\,d\theta$$

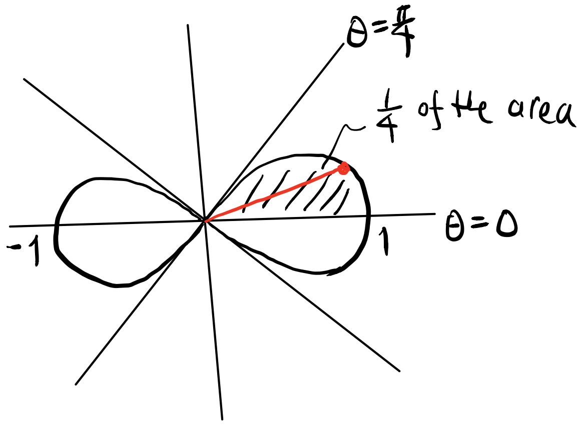
$$= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} r^2\sin\theta\,dr\,d\theta \qquad \#$$

eg 14: Find area enclosed by $r^2 = 4\cos 2\theta$.

Remark: r is "not really" a function of θ, it should be regarded as a "level set".

(i) there is no solution when

$$\frac{\pi}{4}<\theta<\frac{3\pi}{4} \quad \& \quad \frac{5\pi}{4}<\theta<\frac{7\pi}{4}$$



(ii) in term of (x,y) coordinates

$$F(x,y) = (x^2+y^2)^2 - 4(x^2-y^2) = 0 \qquad \text{(check!)}$$

which has a critical point at $(x,y)=(0,0)$ $\left(\vec{\nabla}F(0,0)=\vec{0}\right)$

on the level set. (One cannot apply "Implicit Function Theorem" at the critical point $(0,0)$) (later for more detail)

By the symmetry

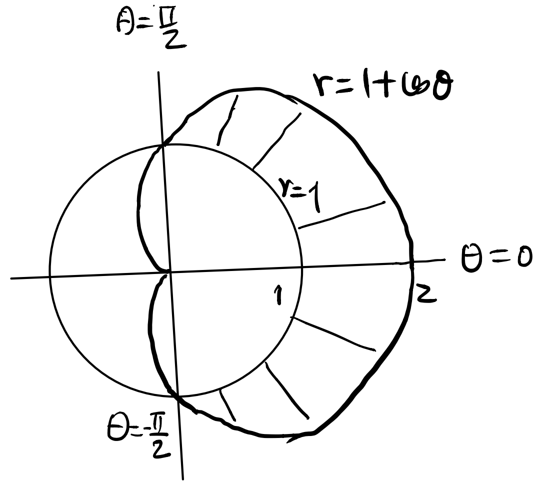
$$\text{Area} = 4\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{4\cos 2\theta}} 1 \cdot r\,dr\,d\theta$$

$$= 8\int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta = 4 \quad \text{(check!)}$$

<u>eg15</u>: Integrate $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ over the region R bounded between

$$\begin{cases} r = 1+\cos\theta & \text{(cardiod)} \\ r = 1 \end{cases}$$

and <u>outside</u> the circle $r=1$



<u>Soln</u>: Intersections:

$1+\cos\theta = 1$

$\Leftrightarrow \cos\theta = 0$

$\Leftrightarrow \theta = \frac{\pi}{2} + k\pi$

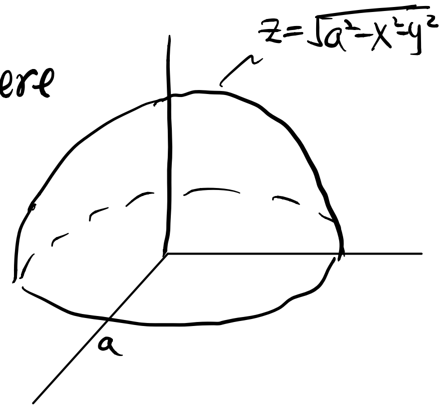
$\therefore \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (choice)

$$\Rightarrow \iint_R f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{1+\cos\theta} \left(\frac{1}{r}\right) \cdot r\,dr\,d\theta$$

$$= 2 \quad \text{(check!)}$$

eg16: Let $z = \sqrt{a^2 - x^2 - y^2}$ be a function defined on

$$R = \{(x,y) : x^2 + y^2 \le a^2\}$$



The graph of z is the (upper) hemisphere of radius a. Find the average height of the hemisphere.

Soln: Average height $= \dfrac{1}{\text{Area}(R)} \iint_R z \, dA$

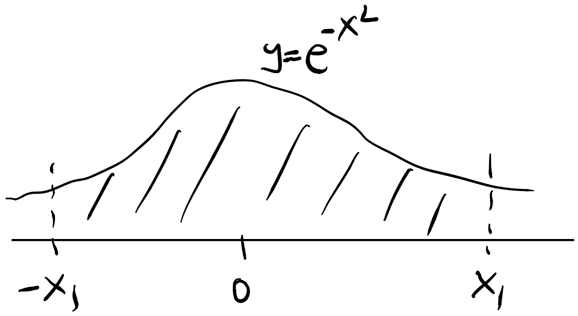
$$= \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^{a} \sqrt{a^2 - r^2} \, r \, dr \, d\theta$$

$$= \frac{2a}{3} \quad \text{(check!)}$$

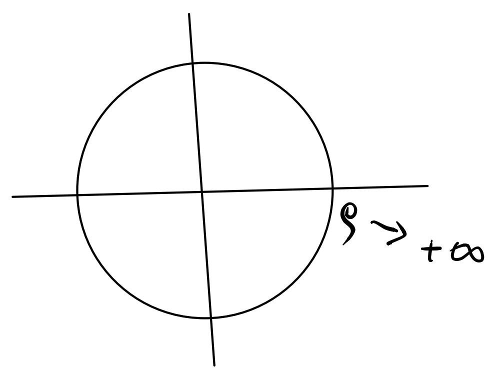
\#

eg17 (Improper integral)

Find $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx$



Soln: Consider $\displaystyle\iint_{\mathbb{R}^2} e^{-x^2-y^2} dA$



$$= \lim_{\rho \to +\infty} \iint_{\{x^2+y^2 \le \rho^2\}} e^{-(x^2+y^2)} dA$$

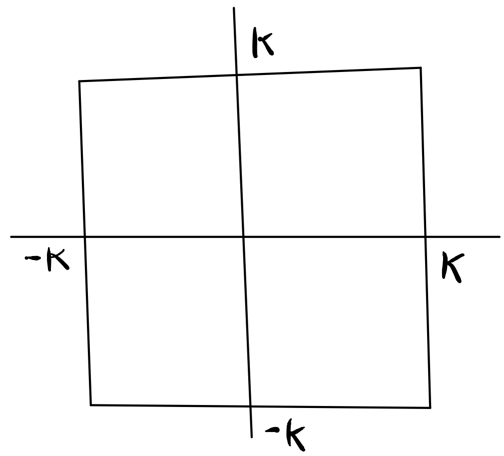
$$= \lim_{\rho \to +\infty} \int_0^{2\pi} \int_0^{\rho} e^{-r^2} \, r \, dr \, d\theta$$

this extra r helps in calculating the integral.

$$= \lim_{\rho \to +\infty} \pi (1 - e^{-\rho^2})$$

$$= \pi$$

On the other hand



$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA$$

$$= \lim_{K \to +\infty} \int_{-K}^{K} \int_{-K}^{K} e^{-x^2-y^2} \, dx \, dy$$

$$= \lim_{K \to +\infty} \int_{-K}^{K} e^{-y^2} \left(\int_{-K}^{K} e^{-x^2} dx \right) dy$$

$$= \lim_{K \to +\infty} \left(\int_{-K}^{K} e^{-x^2} dx \right) \left(\int_{-K}^{K} e^{-y^2} dy \right)$$

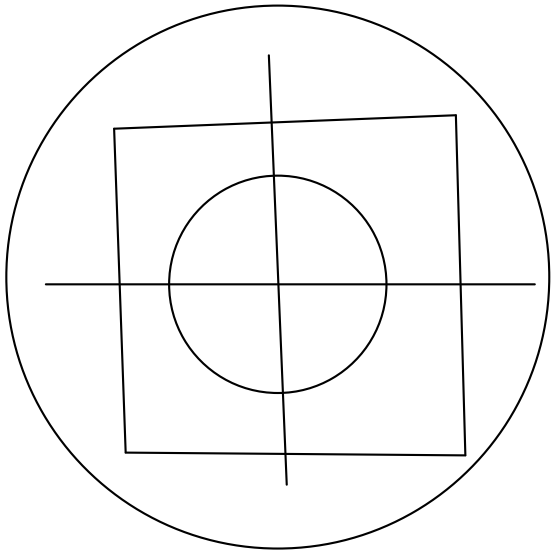
$$= \lim_{K \to +\infty} \left(\int_{-K}^{K} e^{-x^2} dx \right)^2$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2$$

$$\therefore \quad \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} .$$

Caution : we are calculating $\iint_{\mathbb{R}^2} e^{-x^2-y^2} dA$ in <u>two</u> different limiting processes. Why are they equal?

Answer: $e^{-x^2} > 0$ and



Triple Integrals

Def 5 Let $f(x,y,z)$ be a function defined on a (closed and bounded) rectangular box

$$B = [a,b] \times [c,d] \times [r,s]$$

Then the triple integral of f over the box B is

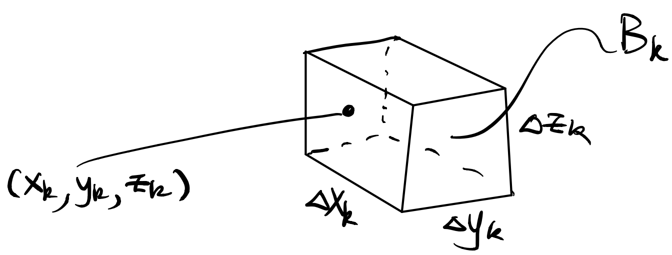
$$\iiint_B f(x,y,z)\,dV = \lim_{\|P\| \to 0} \sum_k f(x_k, y_k, z_k)\,\Delta V_k$$

if this exists

where (i) $P = P_1 \times P_2 \times P_3$ is a subdivision of B into sub-rectangular boxes by partitions P_1, P_2 & P_3 of $[a,b]$, $[c,d]$, and $[r,s]$ respectively. And

$$\|P\| = \max(\|P_1\|, \|P_2\|, \|P_3\|)$$

(ii) (x_k, y_k, z_k) is an arbitrary point in a sub-rectangular box B_k



(iii) $\Delta V_k = \mathrm{Vol}(B_k) = \Delta x_k \Delta y_k \Delta z_k$.

Thm 4 (Fubini's Theorem for Triple Integrals (1st form))

If $f(x,y,z)$ is continuous (in fact, "absolutely" integrable is sufficient) on $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Note: Interchanging the order of the coordinates, we also have

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$$

$= \cdots$ in any order of dx, dy, dz.

Def 6 (Triple integral over a general region $D \subset \mathbb{R}^3$)

Let $f(x,y,z)$ be a function on a closed and bounded region $D \subset \mathbb{R}^3$. Then

$$\iiint_D f(x,y,z)\,dV \overset{\text{def}}{=\!=} \iiint_B F(x,y,z)\,dV$$

where B is a closed and bounded rectangular box containing D, and

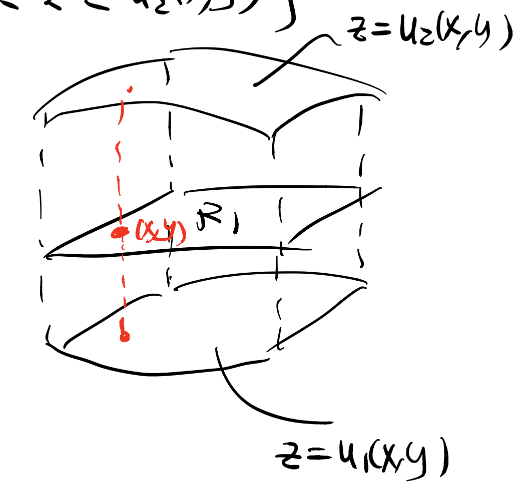
$$F(x,y,z) = \begin{cases} f(x,y,z), & \text{if } (x,y,z) \in D \\ 0, & \text{if } (x,y,z) \in B \setminus D. \end{cases}$$

Note: As in double integral, this definition is well-defined.

Special types of closed and bounded region $D \subset \mathbb{R}^3$

(1) $D = \{(x,y,z) : (x,y) \in R_1,\ u_1(x,y) \le z \le u_2(x,y)\}$

$(u_1(x,y) \le u_2(x,y),\ u_1 \not\equiv u_2)$



(2) $D = \left\{(x,y,z) : \begin{array}{l}(x,z) \in R_2 \\ v_1(x,z) \le y \le v_2(x,z)\end{array}\right\}$

$(v_1 \le v_2,\ v_1 \not\equiv v_2)$

(3) $D = \{(x,y,z) : (y,z) \in R_3,\ w_1(y,z) \le x \le w_2(y,z)\}$

$(w_1 \le w_2,\ w_1 \not\equiv w_2)$

where R_i, $i = 1,2,3$ are closed and bounded plane regions and u_1, u_2; v_1, v_2; w_1, w_2 are continuous wrt the corresponding variables.

Thm 5 (Fubini's Thm for Triple integrals (Strong form))

Let $f(x,y,z)$ be a continuous (absolutely integrable) function on D. If D is of type (1) as above, then

$$\iiint_D f(x,y,z)\, dV = \iint_{R_1} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dx\, dy$$

Similarly for types (2) and (3).

Note: Particularly, we have (using Fubini's for double integrals)

if $D = \left\{ (x,y,z) : \begin{array}{l} a \le x \le b, \quad g_1(x) \le y \le g_2(x) \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}$

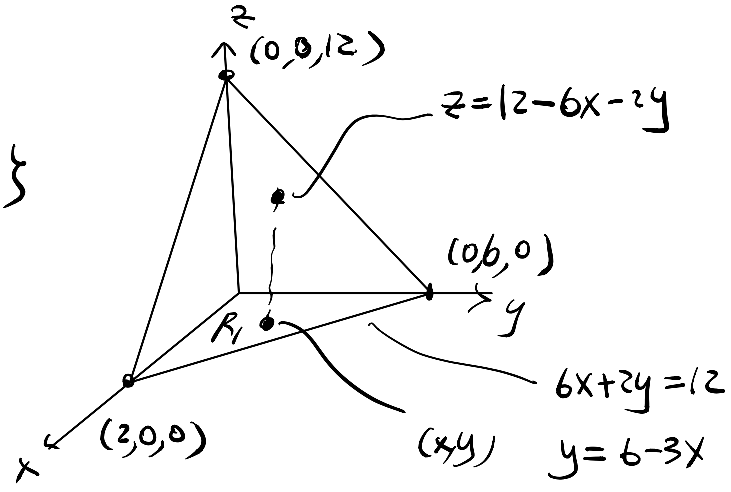
(i.e. R_1 is of type (1) as in double integrals), then

$$\boxed{\iiint_D f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

Similarly for other types.

Prop6: The propositions 1–4 for double integrals also hold for triple integrals over closed and bounded region in $\mathbb{R}^3$.

eg17: Volume of the bounded region D in the 1^{st} octant enclosed by the plane $6x+2y+z=12$



Soln: D is of special type

$= \{ (x,y) \in R_1 : 0 \le z \le 12-6x-2y \}$

$= \left\{ \begin{array}{l} 0 \le x \le 2, \quad 0 \le y \le 6-3x \\ 0 \le z \le 12-6x-2y \end{array} \right\}$

$\Rightarrow \mathrm{Vol}(D) = \iiint_D 1 \cdot dV$

$$= \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} dz\, dy\, dx$$

$$= \cdots = 24 \text{ (check!)}$$

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