Applications

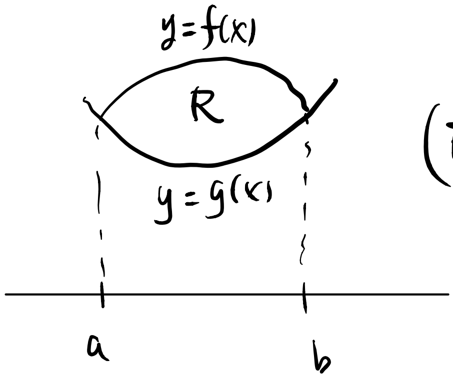
(1) Area (of "good" region $R \subset \mathbb{R}^2$)

$$\boxed{\textbf{Def 3}: \quad \text{Area}(R) = \iint_R 1\, dA}$$

Then Fubini's Thm implies the well-known formula

$$\text{Area}(R) = \int_a^b [f(x) - g(x)]\, dx$$

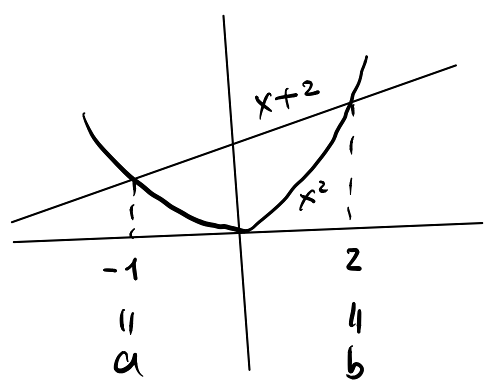
if R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$ ($f(a) = g(a)$, $f(b) = g(b)$, $g(x) \leq f(x)$)



(Ex!)

eg 10 Area bounded by $y = x^2$ and $y = x+2$

Soln: Solving $\begin{cases} y = x^2 \\ y = x+2 \end{cases} \Rightarrow x = -1, 2$



Then by Fubini's

$$\text{Area} = \int_{-1}^{2} (x+2 - x^2)\, dx = \frac{9}{2}$$

(check!)

(2) Average (of a function over a region)

Let $f: R \subset \mathbb{R}^2 \to \mathbb{R}$ be an integrable function

Def 4: The average value of f over R

$$= \frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$

eg 11 Let $f(x,y) = x\cos(xy)$, $R = [0,\pi] \times [0,1]$

Find average of f over R.

Soln: Average of f over $R = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$

$$= \frac{1}{\pi} \int_0^{\pi}\int_0^1 x\cos(xy)\, dy\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin x\, dx \quad \text{(check!)}$$

$$= \frac{2}{\pi} \quad \text{(check!)}$$

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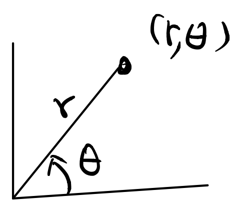
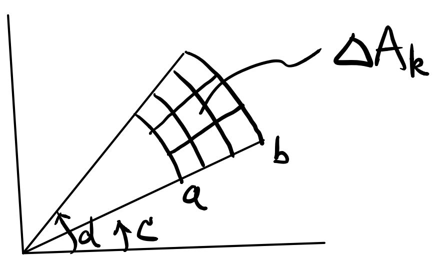
Double integral in polar coordinates



$(r,\theta) \leftrightarrow (x,y)$

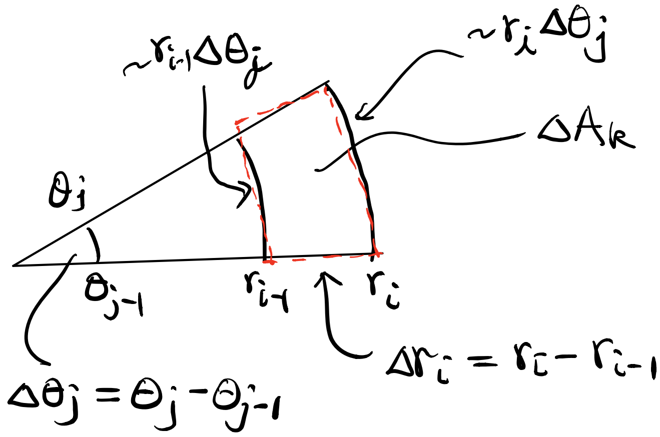
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



$$\begin{cases} a \le r \le b \\ c \le \theta \le d \end{cases}$$

Idea: $\sum_k f(\text{point}_k)\,\Delta A_k$

What is ΔA_k (approximately)?



$\therefore \quad \Delta A_k \simeq (r_i\Delta\theta_j)\cdot\Delta r_i \quad (\simeq (r_{i-1}\Delta\theta_j)\cdot\Delta r_i)$

Hence $\quad \Delta A_k \cong \Delta x\,\Delta y \simeq (r\Delta\theta)\cdot\Delta r$

So $\quad \iint_R f(x,y)\,dA = \iint_R f(x,y)\,\underline{dx\,dy}$

$$= \iint_R f(r\cos\theta, r\sin\theta)\,\underline{r\,dr\,d\theta}$$

Method to remember the formula

$$dA = dx\,dy = r\,dr\,d\theta$$

Double integral of f over $R = \{(r,\theta) : a \le r \le b,\ c \le \theta \le d\}$ in polar coordinates is

$$\iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \int_c^d \left[\int_a^b f(r,\theta)\,r\,dr\right] d\theta$$

$$= \int_a^b \left[\int_c^d f(r,\theta)\,d\theta\right] r\,dr$$

where $f(r,\theta)$ is the simplified notation for $f(r\cos\theta, r\sin\theta)$

Remark: This is a special case of the change of variables formula.

The "extra" factor "r" in the integrand is in fact

$$r = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

the Jacobian determinant of the change of variables.

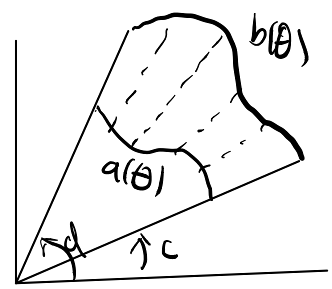
More generally

Thm 3: If R is a (closed and bounded) region with

$c \le \theta \le d$ and

$a(\theta) \le r \le b(\theta)$

$(0 \le a(\theta) \le b(\theta),\ a(\theta) \not\equiv b(\theta))$



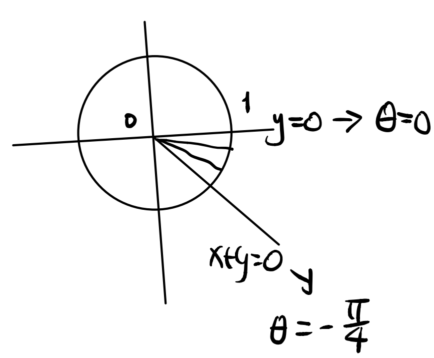
And $f: R \to \mathbb{R}$, then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_{a(\theta)}^{b(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr \right] d\theta$$

(remember this extra "r")

eg 12: Back to our previous example 9

$f(x,y) = x = r\cos\theta$



$$\int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-y}^{\sqrt{1-y^2}} x\,dx\,dy$$

$$= \int_{-\frac{\pi}{4}}^{0} \left[\int_0^1 r\cos\theta \cdot r\,dr \right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{0} \left(\cos\theta \int_0^1 r^2 dr \right) d\theta$$

$$= \cdots = \frac{1}{3\sqrt{2}} \quad (\text{check!})$$

Much easier than before!