Furthermore, we have

Prop3: Let $R = [a,b] \times [c,d]$ be a closed rectangle, $f(x,y)$ and $g(x,y)$ be functions on R, and $k \in \mathbb{R}$ is a constant.

(1) If f & g are integrable over R, then $f \pm g$ and kf are integrable over R.

(2) In the case of (1), we have

$$\iint_R [f \pm g](x,y)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

and

$$\iint_R kf(x,y)\,dA = k \iint_R f(x,y)\,dA.$$

Pf: Omitted (Obvious from the concept of Riemann sum.)

Remark: This Prop 3 implies that the set of integrable functions over (fixed) R forms a "vector space over $\mathbb{R}$", "(double) integral" is linear.

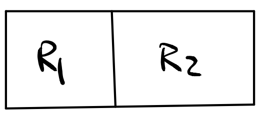
Prop 4: (a) If $f(x,y) \geq 0$ is an integrable function on a closed rectangle R, then
$$\iint_R f(x,y)\,dA \geq 0.$$

(b) If R_1 and R_2 be two closed rectangles such that $\operatorname{int} R_1 \cap \operatorname{int} R_2 = \emptyset$, then
$$\iint_{R_1 \cup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$
for integrable function over $R_1 \cup R_2$.

Pf: Omitted (Obvious from the concept of Riemann sum)

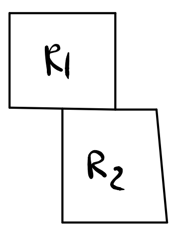
Note: Various situations for $\operatorname{int} R_1 \cap \operatorname{int} R_2 = \emptyset$

(1)

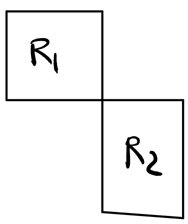


$R_1 \cap R_2$ = common edge
$\operatorname{int} R_1 \cap \operatorname{int} R_2 = \emptyset$

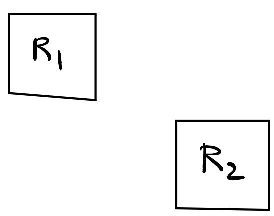
(2)



(3)



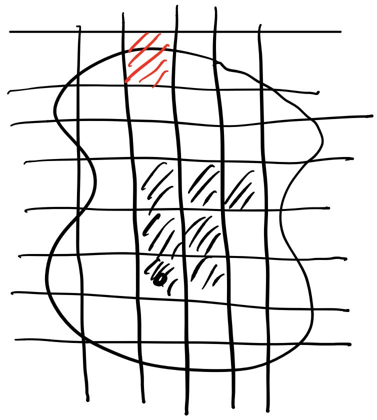
(4)



We haven't define $\iint_{R_1 \cup R_2} f(x,y)\,dA$ for cases (2) - (4)!

Hence we need to define double integrals over general regions.

Double Integrals over General Regions



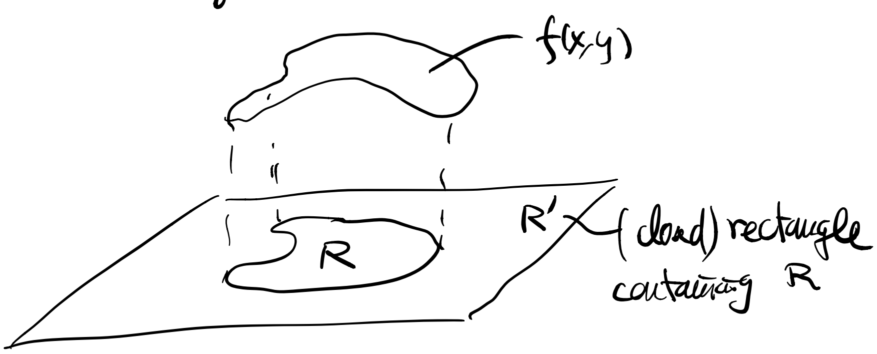
For non-rectangular bounded (closed) region R, one can define similarly the concept of "Riemann sum".

There are two ways to form the "sum"

(i) sum over all subrectangles completely inside R

(ii) sum over all subrectangles with non-empty intersection with R.

Or, one can define "the integrals" as follows



Def 2: Let R be a bounded region and $f(x,y)$ be a function defined on R. For any rectangle $R' \supset R$, define

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in R \\ 0, & (x,y) \in R' \setminus R \end{cases}$$

Then the integral of f over R is defined by

$$\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$

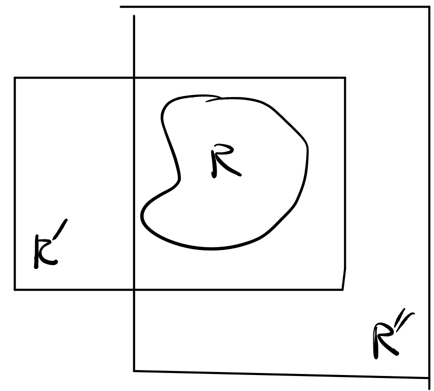
<u>Remark</u>: The definition is <u>well-defined</u> (i.e. doesn't depend on the choice of R'): If R'' is another rectangle s.t. $R'' \supset R$ and

$$\tilde{F}(x,y) = \begin{cases} f(x,y), & (x,y) \in R \\ 0, & (x,y) \in R'' \setminus R \end{cases}$$

Then

$$\iint_{R''} \tilde{F}(x,y)\, dA = \iint_{R'} F(x,y)\, dA$$

(by Prop 4(b))



> **Prop5**: The propositions 1 – 4 hold if we replace "closed rectangle" by "closed and bounded region"

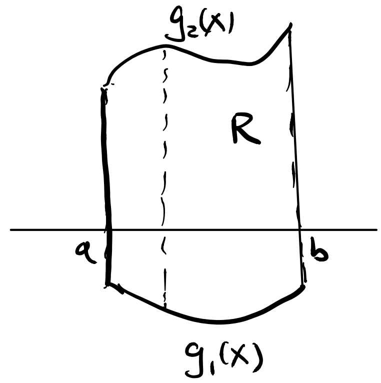
(together with the Prop 2')

<u>Important special types of bounded regions R</u>

Type (1): $R = \{ (x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x) \}$

where g_1 and g_2 are "continuous" functions on $[a,b]$.

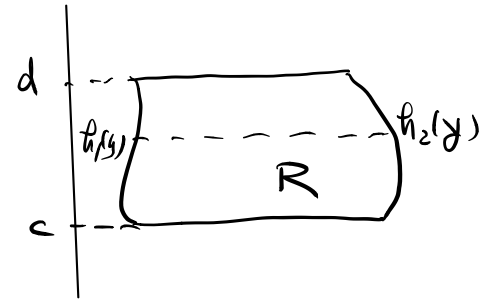
($g_1 \le g_2$, but $g_1 \not\equiv g_2$)



Type (2): $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

where h_1 and h_2 are "continuous" functions on $[c,d]$

($h_1 \le h_2$, but $h_1 \not\equiv h_2$)



For these 2 types of bounded regions, we have

Thm 2 (Fubini's Thm (Stronger version))

Let $f(x,y)$ be a continuous function on a closed and bounded region R.

(1) If R is of type (1) as above, then

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

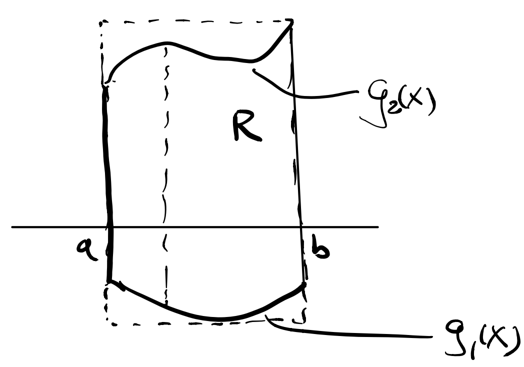
(2) If R is of type (2) as above, then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_{h_1(x)}^{h_2(x)} f(x,y)\,dx \right] dy = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$$

Pf: Type (1): Extend $f(x,y)$ to $F(x,y)$ as in the definition on the rectangle $R' = [a,b] \times [c,d]$ such that

$$c = \min_{[a,b]} g_1(x)$$

$$d = \max_{[a,b]} g_2(x)$$



By definition 2,

$$\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$

$$= \int_a^b \left[\int_c^d F(x,y)\,dy \right] dx \quad \left(\text{Fubini (1}^{\text{st}} \text{ form)} \right)$$

[f continuous on R ⇒ F continuous on R′ except possibly on the boundary (curve(s)) of R. Hence by Prop 2′, F (in fact $|F|$) is integrable over R′. And the Fubini theorem (1st form) is in fact true for "absolutely" integrable functions on a rectangle.]

Now $F(x,y) = 0$ for $y < g_1(x)$ and $y > g_2(x)$,

and $F(x,y) = f(x,y)$ for $g_1(x) \le y \le g_2(x)$.

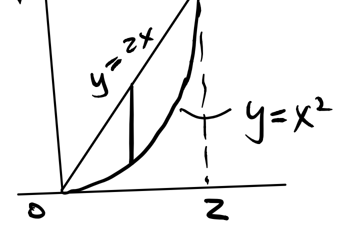
$$\therefore \iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx .$$

Type (2) can be proved similarly. ※

eg 7 Integrate $f(x,y) = 4y + 2$

over the region bounded by $y = x^2$ and $y = 2x$.

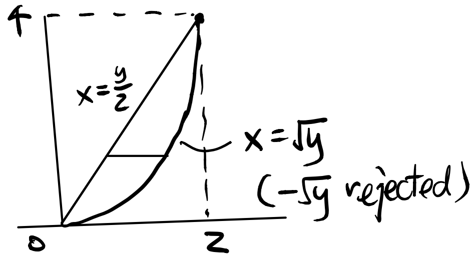
Soln:



need to solve for this point

By Fubini's

$$\iint_R f(x,y)\,dA$$



$$= \int_0^2 \int_{x^2}^{2x} (4y+2)\, dy\, dx$$

$$= \int_0^2 (-2x^4+6x^2+4x)\, dx \quad \text{(check!)}$$

$$= \frac{56}{6} \quad \text{(check!)}$$

In fact, R is also of type (2), and Fubini's

$$\Rightarrow \quad \iint_R f(x,y)\, dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (4y+2)\, dx\, dy$$

$$= \int_0^4 (4y+2)\left(\sqrt{y} - \frac{y}{2}\right) dy$$

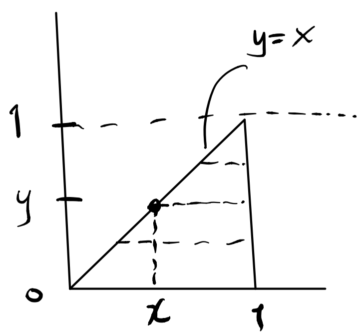
$$= \cdots = \frac{56}{5} \quad \text{(check!)}$$

<u>eg8</u>: Evaluate $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$.

<u>Soln</u>: Regard $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$ as a double integral

of $\frac{\sin x}{x}$ over the region

$y \le x \le 1$ and $0 \le y \le 1$



By Fubini's

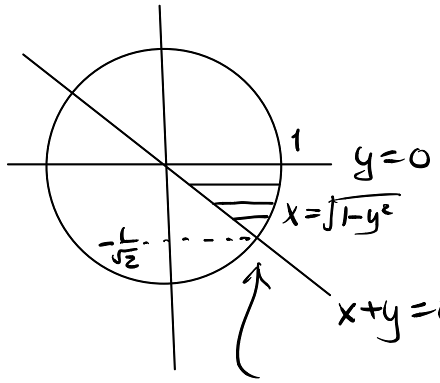
$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \sin x\, dx = 1 - \cos 1$$

(Caution: $f(x,y) = \frac{\sin x}{x}$ doesn't define at $x=0$. why can we use Fubini? ($f(x,y) \ge 0$ & cts except on a line!))

eg 9: Find $\iint_R x\,dA$, where R is the region in the right half-plane bounded by $y=0$, $x+y=0$, and the unit circle.

Soln: Region R (as in the figure)



need to solve eqt. for this part $\begin{cases} x^2+y^2=1 \\ x+y=0 \end{cases}$

$\Rightarrow y=-\frac{1}{\sqrt{2}}$

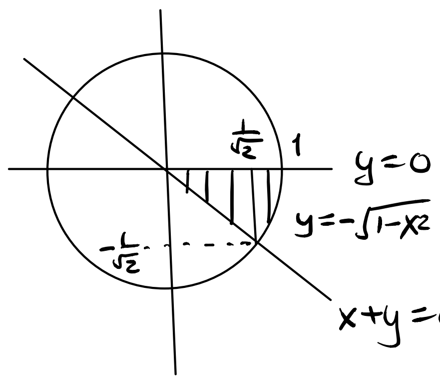
(reject $y=+\frac{1}{\sqrt{2}}$)

By Fubini's

$$\iint_R x\,dA = \int_{-\frac{1}{\sqrt{2}}}^{0}\left(\int_{-y}^{\sqrt{1-y^2}} x\,dx\right)dy$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{0}\left(\frac{1}{2}-y^2\right)dy \text{ (check!)}$$

$$= \frac{1}{3\sqrt{2}} \text{ (check!)}$$



Alternatively,

$$\iint_R x\,dA = \int_0^{\frac{1}{\sqrt{2}}}\left(\int_{-x}^{0} x\,dy\right)dx$$

$$+ \int_{\frac{1}{\sqrt{2}}}^{1}\left(\int_{-\sqrt{1-x^2}}^{0} x\,dy\right)dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x^2\,dx + \int_{\frac{1}{\sqrt{2}}}^{1} x\sqrt{1-x^2}\,dx \text{ (check!)}$$

$$= \frac{1}{3\sqrt{2}} \text{ (check!)}$$