eg5 : let $R=[0,1]\times[0,1]$

$$f(x,y)=\begin{cases} 0, & \text{if both } x \text{ and } y \text{ are rational} \\ 1, & \text{otherwise.} \end{cases}$$

Then f is not integrable over R.

(using (ii))

Soln : $\forall$ (subdivision) partition P of $R=R_1\cup\cdots\cup R_n$

One can find points $(x_k,y_k)\in R_k$, for any k, such that both x_k, y_k are rational. (Why?)

Then corresponding Riemann sum equals

$$S_n(f,P)=\sum_{k=1}^{n} f(x_k,y_k)\Delta A_k=\sum_{k=1}^{n} 0\cdot\Delta A_k=0$$

$$\to 0 \text{ as } \|P\|\to 0.$$

On the other hand, we can also find $(x_k',y_k')\in R_k$ such that at least one of the x_k', y_k' is irrational (why?)

Then corresponding Riemann sum equals

$$S_n'(f,P)=\sum_{k=1}^{n} f(x_k',y_k')\Delta A_k=\sum_{k=1}^{n} 1\cdot\Delta A_k$$

$$= \text{Area}(R)=1$$

$$\to 1 \text{ as } \|P\|\to 0.$$

Since $S_n(f,P)\to 0\neq 1\leftarrow S_n'(f,P)$,

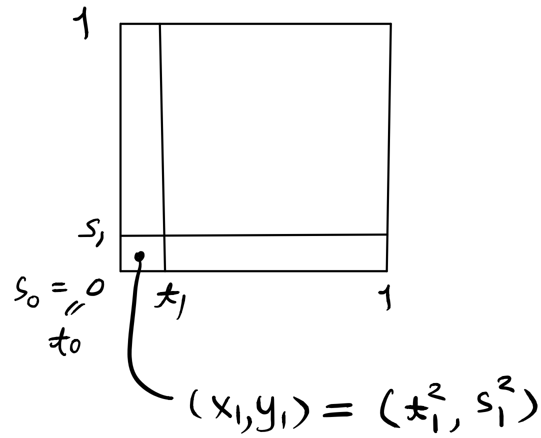
f is not integrable. ※

eg6: Let $R = [0,1]\times[0,1]$

$$f(x,y) = \begin{cases} \frac{1}{xy}, & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0, & \text{if } x = 0 \text{ or } y = 0 \end{cases}$$

Then f is <u>not</u> integrable over R.

(using (i))



Soln: In any partition P of R,

there is a sub-rectangle

$R_1 = [0, t_1]\times[0, s_1]$.

Choose $(x_1, y_1) = (t_1^2, s_1^2) \in R_1$

(since $0 < t_1^2 < t_1 < 1$, $0 < s_1^2 < s_1 < 1$)

Then Riemann sum

$$\begin{aligned} S(f,P) &= \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k \\ &= f(x_1,y_1)\,\Delta A_1 + \underbrace{\sum_{k=2}^{n} f(x_k,y_k)\,\Delta A_k}_{\geq 0} \\ &\geq f(t_1^2, s_1^2)\,\Delta A_1 \\ &= \frac{1}{t_1^2 s_1^2}\cdot t_1 s_1 \\ &= \frac{1}{t_1 s_1} \end{aligned}$$

Since $0 < t_1, s_1 \leq \|P\| \to 0$, $t_1, s_1 \to 0$

Hence $S(f,P) \geq \frac{1}{t_1 s_1} \to \infty$ as $\|P\| \to 0$, i.e. limit doesn't exist.

$\therefore$ f is <u>not</u> integrable. ※

Remark: Egs 5 & 6 show that we need "condition(s)" to ensure the integrability of a function over closed rectangle.

Prop 1: Let $R=[a,b]\times[c,d]$ be a closed rectangle, and $f(x,y)$ be an integrable function over R, then f is bounded on R.

(i.e. $\exists M>0$ such that "$|f(x,y)|\leq M,\ \forall (x,y)\in R$".)

Pf: Omitted (eg6 above gives an idea of proof.)

Remark: From eg5, "boundedness" is necessary, but not sufficient for integrability.

$$\boxed{\text{integrable} \underset{\not\Leftarrow\ (\text{in general})}{\Rightarrow} \text{bounded}}$$

Prop 2: Let $R=[a,b]\times[c,d]$ be a closed rectangle, and $f(x,y)$ be a continuous function on R, then f is integrable on R.

Pf: Omitted (See proof in 1-variable case in MATH2060 for an idea of proof.)

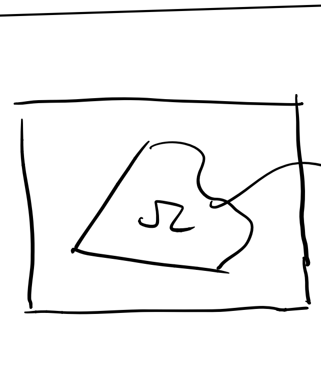
Remarks : (i) Note that a continuous function on closed rectangle is always bounded (Props 1 & 2 are consistent) (MATH 2050 for 1-variable situation)

(ii) "continuity" (on closed rectangle) is sufficient , but not necessary. In fact, Prop 2 can be generalized to a bounded function on a closed rectangle with a "small" set of discontinuity. The precise concept is "measure zero set" (MATH4050 Real Analysis).

For us, we have

> **Prop 2′** For function over closed rectangle
>
> (a) bounded + "continuous except finitely many points"
> $\Rightarrow$ integrable.
>
> (b) bounded + "continuous except finitely many differentiable curves"
> $\Rightarrow$ integrable

eg



$$f(x,y) = \begin{cases} \text{continuous on } \Omega \text{ (and bounded)} \\ 0 \text{ , otherwise (outside } \Omega) \end{cases}$$