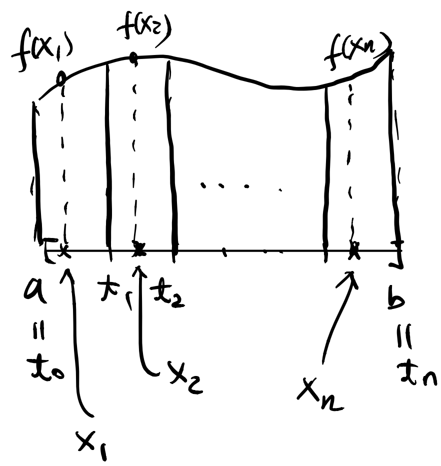
Double Integrals

Recall: In one-variable, "integral" is regarded as "limit" of "Riemann sum" (take MATH 2060 for rigorous treatment)

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

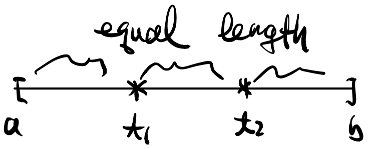
where
- f is a function on the interval $[a,b]$
- P is a <u>partition</u> $a = t_0 < t_1 < t_2 < \cdots < t_n = b$
- $x_k \in [t_{k-1}, t_k]$ and $\Delta x_k = t_k - t_{k-1}$
- $\|P\| = \max_k |\Delta x_k|$



Remark: We usually use <u>uniform partition</u> P

$$a = t_0 < t_1 = a + \frac{1}{n}(b-a) < t_2 = a + \frac{2}{n}(b-a) < \cdots$$

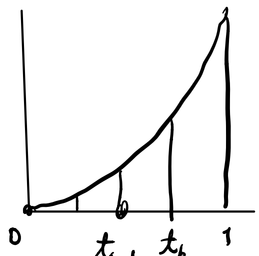
$$\cdots < t_k = a + \frac{k}{n}(b-a) < \cdots = t_n = b$$



In this case, $\|P\| = \max_k |\Delta x_k| = \dfrac{b-a}{n} \to 0$ as $n \to \infty$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot\frac{b-a}{n}$$

eg1 : Find $\int_0^1 x^3 dx$ (i.e. $f(x)=x^3$ on $[0,1]$)



Soln : (1) One may choose

$$x_k = \frac{k-1}{n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

then

$$S_n = \sum_{k=1}^{n} f(x_k)\Delta x_k$$

$$= \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^3 \cdot \frac{1}{n}$$

$$= \frac{1}{n^4} \sum_{k=1}^{n} (k-1)^3$$

$$= \frac{1}{n^4} \cdot \frac{(n-1)^2 n^2}{4} \quad \text{(check!)}$$

$$= \frac{1}{4}\left(1-\frac{1}{n}\right)^2$$

$$\to \frac{1}{4} \quad \text{as } n \to \infty$$

$$\therefore \quad \int_0^1 x^3\, dx = \frac{1}{4}$$

(2) Or, we can choose $x_k = \frac{k}{n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$

(Will we get different answer ?)

Then

$$S_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \cdot \frac{1}{n}$$

$$= \frac{1}{n^4} \cdot \frac{n^2 (n+1)^2}{4}$$

$$= \frac{1}{4}\left(1+\frac{1}{n}\right)^2$$

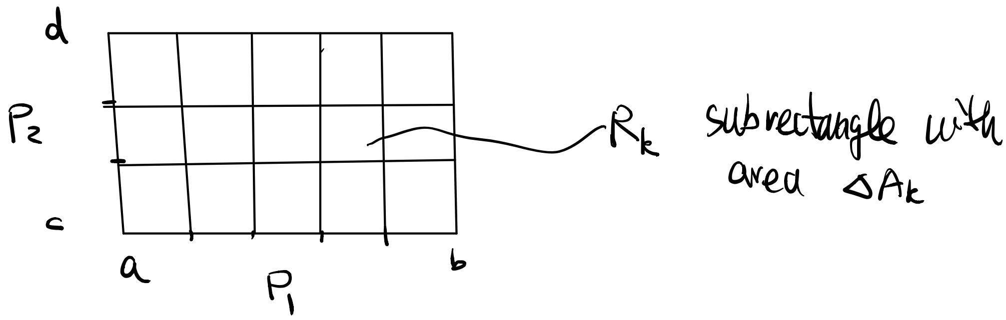
$$\to \frac{1}{4} \quad \text{as } n \to \infty$$

(Same limit)

Remark : We can use any $x_k \in [t_{k-1}, t_k]$ and still get the same $\int_0^1 x^3 dx = \frac{1}{4}$.

This concept can be generalized to <u>any dimension</u>.

For 2-dim., let we first consider a function $f(x,y)$ defined on a <u>rectangle</u> $R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$



Then we can subdivide R into sub-rectangles by using partitions P_1 of $[a,b]$ & P_2 of $[c,d]$.

Denote $P = P_1 \times P_2$ (partition, subdivision, of R)

and $\|P\| = \max(\|P_1\|, \|P_2\|)$

Let the sub-rectangles be R_k, $k = 1, \cdots, N$ = number of subrectanges

with <u>areas</u> ΔA_k

Choose point $(x_k, y_k) \in R_k$ (for each $k = 1, \cdots, N$), then consider the sum

$$S(f,P) = \sum_{k=1}^{N} f(x_k, y_k)\, \Delta A_k$$

Def 1: The function f is said to be <u>integrable</u> over R

if $\lim_{\|P\| \to 0} S(f, P) = \lim_{\|P\| \to 0} \sum_{k=1}^{N} f(x_k, y_k) \Delta A_k$

<u>exists</u> and <u>independent</u> of the choice of $(x_k, y_k) \in R_k$.

In this case, the limit is called the (<u>double</u>) <u>integral</u> <u>of f over R</u> and is denoted by

$$\iint_R f(x,y)\, dA \quad \text{or} \quad \iint_R f(x,y)\, dx\, dy$$

<u>Remark</u>: Same as 1-variable, the double integral of f, $f \geq 0$, over R can be interpreted as <u>volume</u> under the graph of f (over R)

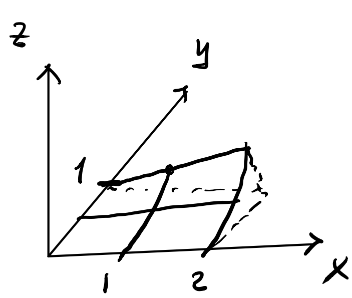
volume $\sim f(x_k, y_k) \Delta A_k$

- And when $f \equiv 1$,

$$\iint_R 1\, dA \text{ is the area of } R.$$

<u>eg 2</u>: $R = [0,2] \times [0,1]$, $f(x,y) = xy^2$

(using definition) Find $\iint_R xy^2\, dx\, dy$



<u>Soln</u>: Using the uniform partitions:

$P_1 = \{0, \frac{2}{n}, \frac{4}{n}, \cdots, 2\}$ of $[0,2]$

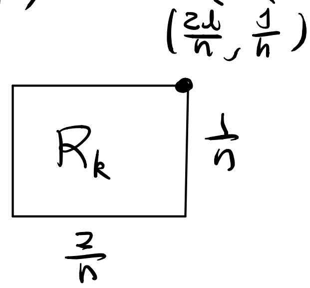
$P_2 = \{0, \frac{1}{n}, \frac{2}{n}, \cdots 1\}$ of $[0,1]$

$\Rightarrow$ a particular sub-rectangle is

$$R_k = \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$$

for some $i = 1, \cdots, n$, $j = 1, \cdots, n$.

(So R_k should be better denoted by R_{ij})



(Assume it is integrable)

One may choose the point

$$(x_k, y_k) = \left(\frac{2i}{n}, \frac{j}{n}\right) \in R_k$$

and consider the Riemann sum

$$\sum_k f(x_k, y_k)\, \Delta A_k$$

$$= \sum_{i,j=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{j}{n}\right)^2 \cdot \frac{2}{n} \cdot \frac{1}{n}$$

$$= \frac{4}{n^5} \sum_{i,j=1}^{n} i j^2$$

$$= \frac{4}{n^5} \sum_{i=1}^{n} \left[i \sum_{j=1}^{n} j^2 \right]$$

$$= \frac{4}{n^5} \left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{n} j^2\right)$$

$$= \frac{4}{n^5} \cdot \frac{n(n+1)}{2} \cdot \frac{n(n+1)(2n+1)}{6} \to \frac{4 \cdot 2}{2 \cdot 6} = \frac{2}{3} \text{ as } n \to \infty$$

$\therefore \displaystyle\iint_{[0,2]\times[0,1]} xy^2\, dx\, dy = \frac{2}{3}$ ※ Very tedious calculation.

Hence we need the following Theorem:

Thm 1 (Fubini's Theorem (1st form))

If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

The last 2 integrals above are called iterated integrals

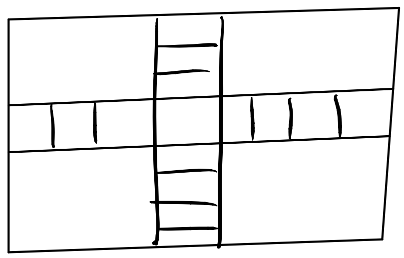
(Pf: Omitted)

Ideas:

sum horizontally first & taking limit

$\rightarrow \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$

("$\sum_i$" 1st in eg 2)



sum vertically first & taking limit

$\rightarrow \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$

("$\sum_j$" 1st in eg 2)

eg 3: Using Fubini to calculate $\iint_R xy^2\,dx\,dy$, $R=[0,2]\times[0,1]$

Soln: By Fubini

$$\iint_R xy^2\,dA = \int_0^2 \left(\int_0^1 xy^2\,dy\right) dx$$

$$= \int_0^2 \left(x \int_0^1 y^2\,dy\right) dx$$

$$= \int_0^2 \frac{x}{3}\,dx$$

$$= \frac{2}{3}$$

or $\iint_R xy^2 dA = \int_0^1 \left(\int_0^2 xy^2 dx \right) dy$

$$= \int_0^1 \left(y^2 \int_0^2 dx \right) dy$$

$$= \int_0^1 2y^2 dy$$

$$= \frac{2}{3}$$

Much easier than using Riemann sum ! #

eg4 : Some times the "order" of the iterated integrals is important in practice :

Find $\displaystyle\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA$.

Soln : $\displaystyle\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^{\pi} \left[\int_0^1 x\sin(xy)\,dx \right] dy$

$$= \int_0^{\pi} \left(-\frac{\cos y}{y} + \frac{\sin y}{y^2} \right) dy$$ (check! use integration-by-parts)

Not easy to integrate !

On the other hand, in different order

$$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^1 \left[\int_0^{\pi} x\sin(xy)\,dy \right] dx$$

$$= \int_0^1 (-\cos\pi x + 1)\,dx$$

$$= 1 \quad \text{(easy!)}$$

#

Caution: Not all functions are integrable over a (closed) rectangle.

Remark:
- To show "integrable", needs to show that for all partitions and for all points (x_k, y_k) in the subrectangles, the Riemann sum

$$S(f, P) \to \text{the same number (as } \|P\| \to 0)$$

- To disprove "integrable", needs to find, for examples

(i) some P with some choose of (x_k, y_k) such that

$$\lim_{\|P\| \to 0} S(f, P) \text{ doesn't exist.}$$

(ii) some P with different (x_k, y_k) & (x_k', y_k') such that

$$\underset{\text{with } (x_k, y_k)}{S(f,P)} \to a \neq b \leftarrow \underset{\text{with } (x_k', y_k')}{S'(f,P)}$$

eg5: Let $R = [0,1] \times [0,1]$

$$f(x,y) = \begin{cases} 0, & \text{if both } x \text{ and } y \text{ are rational} \\ 1, & \text{otherwise.} \end{cases}$$

Then f is not integrable over R.

(using (ii))

eg6: Let $R = [0,1] \times [0,1]$

$$f(x,y) = \begin{cases} \frac{1}{xy}, & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0, & \text{if } x = 0 \text{ or } y = 0 \end{cases}$$

Then f is <u>not</u> integrable over R.

(using (i))