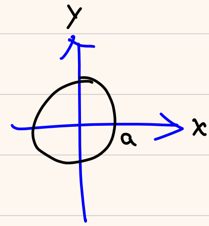
Math 2020B Tut 6

Q1: Find the perimeter of the circle $x^2+y^2=a^2$ $(a>0)$.



Ans: Step 1: Find a parametrization,

$$\gamma: [0,2\pi] \to \mathbb{R}^2, \quad \gamma(t) = (a\cos t, a\sin t)$$

Step 2: $\int_\gamma ds = \int_0^{2\pi} |\gamma'(t)|\,dt$

We compute: $\gamma'(t) = (-a\sin t, a\cos t)$

$$\Rightarrow |\gamma'(t)| = \int_0^{2\pi} a\,dt = 2\pi a$$

Q2: Let $f:[0,\sqrt{2}] \to \mathbb{R}$, $f(x) = x^2$. $g:\mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = x$

Find the integral of g along the curve $\gamma: t \mapsto (t, f(t))$, $t \in [0,\sqrt{2}]$

Ans: $\int ds = \int_0^{\sqrt{2}} g(\gamma(t))|\gamma'(t)|\,dt$.

Now, $\gamma(t) = (t, f(t))$

$\gamma'(t) = (1, f'(t)) = (1, 2t)$, $|\gamma'(t)| = \sqrt{1+4t^2}$

$g(\gamma(t)) = g(t, f(t)) = t$

So $\int_0^{\sqrt{2}} g(\gamma(t))|\gamma'(t)| = \int_0^{\sqrt{2}} t\sqrt{1+4t^2}\,dt$

$$= \frac{1}{12}(1+4t^2)^{\frac{3}{2}}\Big|_0^{\sqrt{2}}$$

$$= \frac{13}{6}$$

Q3: Let $\gamma : [a,b] \to \mathbb{R}^2$, with $\gamma(a) = \vec{p}$, $\gamma(b) = \vec{q}$.

Assume γ is differentiable, show that the length of γ is $\geq |\vec{q} - \vec{p}|$

Ans: Cauchy schwarz: $|\gamma'(t)||\vec{q}-\vec{p}| \geq \langle \gamma'(t), \vec{q}-\vec{p} \rangle$

Therefore, $\text{length}(\gamma) = \int_a^b |\gamma'(t)| dt$

$$\geq \int_a^b \langle \gamma'(t), \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} \rangle dt$$

If we define, $F : \mathbb{R} \to \mathbb{R}$

$$F(t) = \langle \gamma(t), \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} \rangle$$

Then we have $F'(t) = \langle \gamma'(t), \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} \rangle$

Hence $\text{length}(\gamma) \geq \int_a^b F'(t) dt$

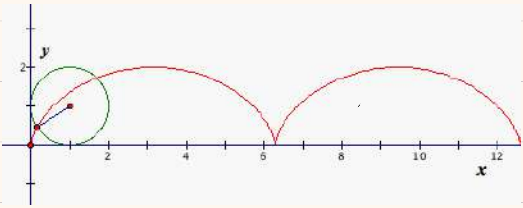
$$= F(b) - F(a)$$

$$= \langle \gamma(b), \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} \rangle - \langle \gamma(a), \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} \rangle$$

$$= \langle \vec{q}-\vec{p}, \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} \rangle$$

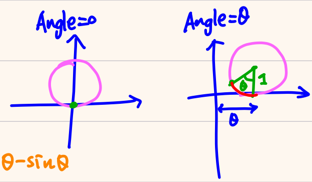
$$= |\vec{q}-\vec{p}|$$

Q 4:



A circle of radius 1 in the xy-plane rolls without slipping along x-axis.
The red line is the trace of a point on the circle.
Find the length for the part of the trace corresponding to a complete rotation of the circle.

Ans: Step 1: Find a parametrization



x = length of the red arc $- \sin\theta = \theta - \sin\theta$

$y = 1 - \cos\theta$

Thus $\gamma: [0, 2\pi] \to \mathbb{R}^2$, $\gamma(\theta) = (\theta - \sin\theta,\ 1 - \cos\theta)$

Step 2: $\gamma'(\theta) = (1 - \cos\theta,\ \sin\theta)$

$|\gamma'(\theta)| = \sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2 - 2\cos\theta} = 2\sin\theta/2$ (which ≥ 0 for $\theta \in [0, 2\pi]$)

Therefore, length $= \int_0^{2\pi} 2\sin\theta/2 \, d\theta$

$= -4\cos\theta/2 \Big|_0^{2\pi}$

$= 8$