MATH2020A Tutorial 8

1. Calculate the length of cycloid defined by $\gamma(t) = (x(t), y(t))$ with

$$\begin{cases} x(t) = & t - \sin t \\ y(t) = & 1 - \cos t \end{cases} \quad t \in [0, 2\pi]$$

$$\begin{aligned} L(\gamma) &= \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t} dt \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos t} dt = \int_0^{2\pi} 2|\cos \frac{t}{2}| dt = 4 \int_0^{\pi} \cos \frac{t}{2} dt = 8 \end{aligned}$$

Remark: This curve has a lot of useful properties in physics.
2. Try to calculate the perimeter of ellipse defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Using $e = \frac{\sqrt{a^2 - b^2}}{a}$ to express integration.
We can describe this curve by the following parameterization.

$$\begin{cases} x(t) = & a \cos t \\ y(t) = & b \sin t \end{cases} \quad t \in [0, 2\pi]$$

So we get

$$\begin{aligned} L(\gamma) &= \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt \\ &= \int_0^{2\pi} \sqrt{a^2(1 - \cos^2 t) + b^2 \sin^2 t} dt = \int_0^{2\pi} a\sqrt{1 - e^2 \cos^2 t} dt \\ &= \int_0^{2\pi} a\sqrt{1 - e^2 \sin^2 t} dt \end{aligned}$$

We cannot continuous, because we don't know how to calculate the last integration using traditional method. This integration is called the second kind of elliptic integration.

3. Line integral of Vector Fields. Define $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$. Find the integration of this vector fields along the squre $\{(x,y)|\ |x| = 1 \text{ or } |y| = 1 \text{ with } |x|,|y| \leq 1\}$.

This curve contains four straight lines. $\mathrm{r}_1(t) = \mathbf{i} + t\mathbf{j}$, $\mathrm{r}_2(t) = -t\mathbf{i} + \mathbf{j}$, $\mathrm{r}_3(t) = -\mathbf{i} - t\mathbf{j}$, $\mathrm{r}_4(t) = t\mathbf{i} - \mathbf{j}$, $t \in [-1,1]$. Hence

$$\begin{aligned}
\int_{C_1} \frac{-ydx + xdy}{x^2+y^2} &= \int_{-1}^{1} \frac{dy}{1+y^2} = \frac{\pi}{2} \\
\int_{C_2} \frac{-ydx + xdy}{x^2+y^2} &= \int_{-1}^{1} \frac{dx}{x^2+1} = \frac{\pi}{2} \\
\int_{C_3} \frac{-ydx + xdy}{x^2+y^2} &= \int_{-1}^{1} \frac{dy}{1+y^2} = \frac{\pi}{2} \\
\int_{C_4} \frac{-ydx + xdy}{x^2+y^2} &= \int_{-1}^{1} \frac{dx}{x^2+1} = \frac{\pi}{2}
\end{aligned}$$

In summary, we have

$$\int_C \frac{-ydx + xdy}{x^2+y^2} = 2\pi$$

4. Calculating the following integrals.

$$\int_C xz^2dx + yx^2dy + zydz$$

with $C : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, t \in [0,1]$.

$$\begin{aligned}
\int_C xz^2dx + yx^2dy + zydz &= \int_0^1 tt^6dt + t^2t^2(2tdt) + t^3t^2(3t^2dt) \\
&= \int_0^1 (4t^7 + 2t^5)dt \\
&= \frac{5}{6}
\end{aligned}$$

5. Find the following integration

$$\int_C x^2ds$$

along the curve intersected by sphere $x^2+y^2+z^2=1$ and plane $x+y=0$.

Changing variables, $\tilde{x} = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$ and $\tilde{y} = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$. Then the curve becomes the intersection of

$$\tilde{x}^2 + \tilde{y}^2 + z^2 = 1 \quad \text{and} \quad \tilde{x} = 0$$

So we can present this curve by

$$\mathbf{r}(t) = 0\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$$

Solve x will give us $x = \frac{\tilde{x}}{\sqrt{2}} + \frac{\tilde{y}}{\sqrt{2}}$. Hence

$$\begin{aligned}
\int_C x^2 ds &= \int_0^{2\pi} \frac{1}{2}(\tilde{x}+\tilde{y})^2 ds = \int_0^{2\pi} \frac{\cos^2 t}{2}\sqrt{\cos'(t)^2 + \sin'(t)^2} dt \\
&= \int_0^{2\pi} \frac{\cos^2 t}{2} dt = \frac{\pi}{2}
\end{aligned}$$

What if we change the curve by the intersection of sphere $x^2+y^2+z^2=1$ and plane $x+y+z=0$ to find out the integration?

Change variables according following formula.

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

So we find $\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 1$ and $\tilde{x} = 0$. Using the property of orthogonal matrix, we have $x = \frac{1}{\sqrt{3}}\tilde{x} + \frac{1}{\sqrt{2}}\tilde{y} + \frac{1}{\sqrt{6}}\tilde{z}$. Hence

$$\int_C x^2 ds = \int_0^{2\pi} \left(\frac{\cos t}{\sqrt{2}} + \frac{\sin t}{\sqrt{6}}\right)^2 dt = \int_0^{2\pi} \frac{\cos^2 t}{2} + \frac{\sin^2 t}{6} + \frac{2\cos t \sin t}{2\sqrt{3}} dt = \frac{2}{3}\pi$$