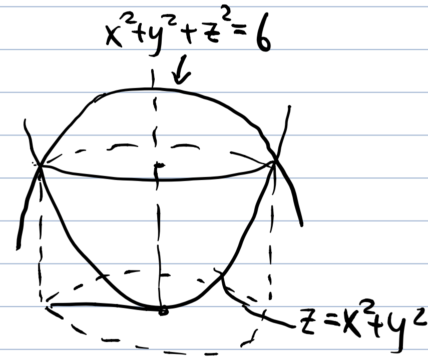
Version 1

1. Intersection of $z = x^2+y^2$ and $x^2+y^2+z^2=6$ :




$$\begin{cases} z = x^2+y^2 \\ x^2+y^2+z^2=6 \end{cases}$$

$$\Rightarrow \quad z^2+z=6 \quad \Rightarrow \quad z = \frac{-1+\sqrt{1+24}}{2} = 2 \qquad \left(\begin{array}{c}\text{since } z=x^2+y^2 \\ \text{facing upward}\end{array}\right)$$

$\Rightarrow$ radius of the projected disk $= \sqrt{x^2+y^2} = \sqrt{z} = \sqrt{2}$

Using cylindrical coordinates,

$$\text{Vol. of the solid} = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{\sqrt{6-r^2}} r\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^{\sqrt{2}} r \cdot \left[\sqrt{6-r^2} - r^2\right] dr$$

$$= \pi \int_0^2 \left(\sqrt{6-t} - t\right) dt \qquad (\text{by letting } t = r^2)$$

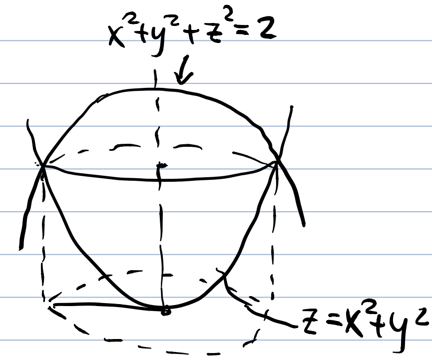
$$= \pi \left[ -\frac{2(6-t)^{3/2}}{3} - \frac{t^2}{2} \right]_0^2$$

$$= \pi \left[ \left( -\frac{2}{3} 4^{3/2} - 2 \right) + \frac{2}{3} 6^{3/2} \right]$$

$$= \frac{12\sqrt{6} - 22}{3} \pi \quad \#$$

Version 2

1. Intersection of $z=x^2+y^2$ and $x^2+y^2+z^2=2$ :




$$\begin{cases} z=x^2+y^2 \\ x^2+y^2+z^2=2 \end{cases}$$

$\Rightarrow \quad z^2+z=2 \quad \Rightarrow \quad z=\dfrac{-1+\sqrt{1+8}}{2}=1$ (since $z=x^2+y^2$ facing upward)

$\Rightarrow$ radius of the projected disk $=\sqrt{x^2+y^2}=\sqrt{z}=1$

Using cylindrical coordinates,

$$\text{Vol. of the solid} = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^1 r\cdot\left[\sqrt{2-r^2}-r^2\right]dr$$

$$= \pi\int_0^1\left(\sqrt{2-t}-t\right)dt \quad (\text{by letting } t=r^2)$$

$$= \pi\left[-\frac{2(2-t)^{3/2}}{3}-\frac{t^2}{2}\right]_0^1$$

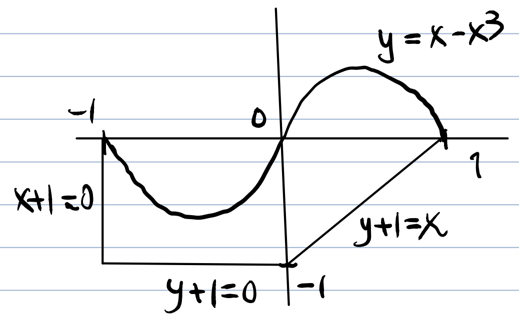
$$= \pi\left[\left(-\frac{2}{3}-\frac{1}{2}\right)+\frac{2}{3}2^{3/2}\right]$$

$$= \frac{8\sqrt{2}-7}{6}\pi$$

$\#$

Version 1

2 (a) $R = \{ x+1 \geq 0,\ y+1 \geq 0,\ y+1 \geq x \ \&\ x - x^3 = y \}$




(b) $$\iint_R x\,dA = \int_{-1}^{0}\int_{-1}^{x-x^3} x\,dy\,dx + \int_0^1 \int_{x-1}^{x-x^3} x\,dy\,dx$$

$$= \int_{-1}^{0} x\left[(x-x^3)+1\right]dx + \int_0^1 x\left[(x-x^3)-(x-1)\right]dx$$

$$= \int_{-1}^{0}(x^2 - x^4 + x)\,dx + \int_0^1 (-x^4 + x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^2}{2}\right]_{-1}^{0} + \left[-\frac{x^5}{5} + \frac{x^2}{2}\right]_0^1$$

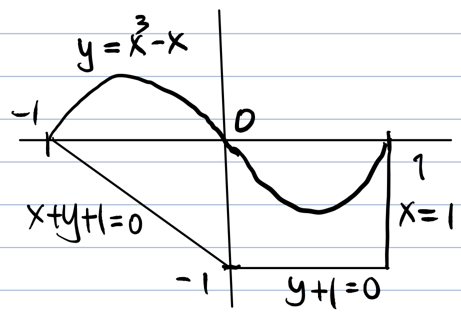
$$= -\left(-\frac{1}{3} + \frac{1}{5} + \frac{1}{2}\right) + \left(-\frac{1}{5} + \frac{1}{2}\right)$$

$$= \frac{1}{3} - \frac{2}{5}$$

$$= \frac{-1}{15}$$ #

Version 2

2 (a) $R = \{ 1 \geq x \ , y+1 \geq 0, \ x+y+1 \geq 0 \ \& \ x^3 - x = y \}$




(b) $$\iint_R x\,dA = \int_{-1}^{0}\int_{-(1+x)}^{x^3-x} x\,dy\,dx + \int_0^1 \int_{-1}^{x^3-x} x\,dy\,dx$$

$$= \int_{-1}^{0} x\left[(x^3-x)+(1+x)\right]dx + \int_0^1 x\left[(x^3-x)+1\right]dx$$

$$= \int_{-1}^{0} (x^4 + x)\,dx + \int_0^1 (x^4 - x^2 + x)\,dx$$

$$= \left[\frac{x^5}{5} + \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^5}{5} - \frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$$

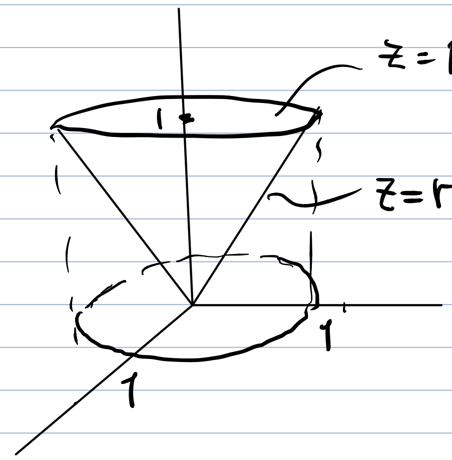
$$= -\left(-\frac{1}{5} + \frac{1}{2}\right) + \left(\frac{1}{5} - \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{2}{5} - \frac{1}{3}$$

$$= \frac{1}{15}$$ #

Version 1

3. $$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^{1} x^2z^2\,dz\,dx\,dy$$




$$= \int_0^{2\pi}\int_0^1\int_r^1 (x^2z^2)\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_r^1 z^2 r^3 \cos^2\theta\, dz\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} \cos^2\theta\, d\theta\right)\left(\int_0^1 r^3 \,\frac{1-r^3}{3}\, dr\right)$$

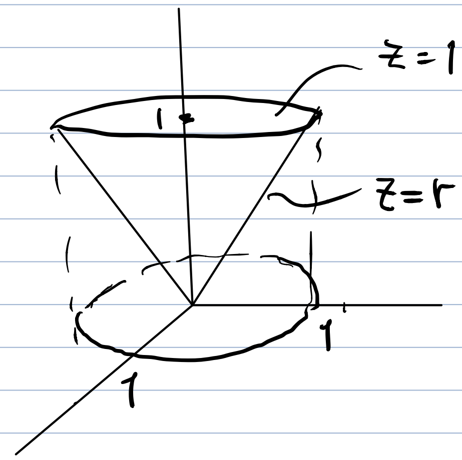
$$= \int_0^{2\pi} \frac{1}{2}(1+\cos 2\theta)\, d\theta \cdot \frac{1}{3}\int_0^1 (r^3 - r^6)\,dr$$

$$= \pi \cdot \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right)$$

$$= \frac{\pi}{28}$$

Version 2

3. $$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^{1} x^2z^3\,dz\,dx\,dy$$




$$= \int_0^{2\pi}\int_0^1\int_r^1 (x^2z^3)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_r^1 z^3 r^3 \cos^2\theta\,dz\,dr\,d\theta$$

$$= \left(\int_0^{2\pi}\cos^2\theta\,d\theta\right)\left(\int_0^1 r^3\cdot\frac{1-r^4}{4}\,dr\right)$$

$$= \int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)\,d\theta \cdot \frac{1}{4}\int_0^1 (r^3-r^7)\,dr$$

$$= \pi\cdot\frac{1}{4}\left(\frac{1}{4}-\frac{1}{8}\right)$$

$$= \frac{\pi}{32}$$

Version 1

4. Let $u = x$
$v = ax + by + cz$
$w = ax + cy - bz$

Then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{pmatrix} 1 & 0 & 0 \\ a & b & c \\ a & c & -b \end{pmatrix} = -(b^2+c^2) < 0$

$\Rightarrow \quad \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{-1}{b^2+c^2}$

Change of variables famula $\Rightarrow$

$$\iiint_D x^2(ax+by+cz)^6(ax+cy-bz)^4\,dV$$

$$= \int_0^\alpha \int_0^\beta \int_0^\gamma u^2 v^6 w^4 \left|\frac{-1}{b^2+c^2}\right| du\,dv\,dw$$

$$= \frac{1}{b^2+c^2} \int_0^\alpha \int_0^\beta \int_0^\gamma u^2 v^6 w^4\, du\,dv\,dw$$

which is independent of $a$ #

$$\left( = \frac{1}{b^2+c^2} \cdot \frac{\alpha^3 \beta^7 \gamma^5}{3 \cdot 7 \cdot 5} \right)$$

Version 2

4. Let $u = x$

$v = ax - by + cz$

$w = ax + cy + bz$

Then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{pmatrix} 1 & 0 & 0 \\ a & -b & c \\ a & c & b \end{pmatrix} = -(b^2+c^2) < 0$

$\Rightarrow \quad \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{-1}{b^2+c^2}$

Change of variables formula $\Rightarrow$

$$\iiint_D x^2 (ax-by+cz)^6 (ax+cy+bz)^4 \, dV$$

$$= \int_0^\alpha \int_0^\beta \int_0^\gamma u^3 v^7 w^5 \left| \frac{-1}{b^2+c^2} \right| du\,dv\,dw$$

$$= \frac{1}{b^2+c^2} \int_0^\alpha \int_0^\beta \int_0^\gamma u^3 v^7 w^5 \, du\,dv\,dw$$

which is independent of $a$ ※

$$\left( = \frac{1}{b^2+c^2} \cdot \frac{\alpha^3 \beta^7 \gamma^5}{3 \cdot 7 \cdot 5} \right)$$

# Version 1

$$I_n=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty}(x^2+y^2+z^2)^{\frac{n}{2}}e^{-2(x^2+y^2+z^2)}\,dx\,dy\,dz$$

$$=\lim_{r\to+\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{r}\rho^n e^{-2\rho^2}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\lim_{r\to+\infty}2\pi\int_0^{\pi}\sin\phi\,d\phi\int_0^{r}\rho^{n+2}e^{-2\rho^2}d\rho$$

$$=\lim_{r\to+\infty}4\pi\int_0^{r}\rho^{n+2}e^{-2\rho^2}d\rho$$

$$=4\pi\lim_{r\to+\infty}\int_0^{r}\rho^{n+2}\frac{de^{-2\rho^2}}{-4\rho}$$

$$=-\pi\lim_{r\to+\infty}\int_0^{r}\rho^{n+1}\,de^{-2\rho^2}$$

$$=-\pi\lim_{r\to+\infty}\left[\rho^{n+1}e^{-2\rho^2}\Big|_0^r-\int_0^r e^{-2\rho^2}d\rho^{n+1}\right]$$

$$=-\pi\lim_{r\to+\infty}\left(r^{n+1}e^{-2r^2}-(n+1)\int_0^r\rho^n e^{-2\rho^2}d\rho\right)$$

$$=\frac{(n+1)}{4}\lim_{r\to+\infty}4\pi\int_0^r\rho^n e^{-2\rho^2}d\rho$$

$$\therefore\quad I_n=\frac{n+1}{4}I_{n-2}$$

$$\Rightarrow \quad I_n = \frac{n+1}{4} \cdot \frac{(n-1)}{4} \cdots \frac{(n-2k+3)}{4} \cdot I_{n-2k}$$

$$= \frac{(n+1)(n-1)\cdots(n-2k+3)}{4^k} I_{n-2k}$$

$$= \begin{cases} \dfrac{(n+1)(n-1)\cdots 3\cdot 1}{4^{\frac{n}{2}+1}} I_{-2} & \text{if } n = \text{even} \\[2ex] \dfrac{(n+1)(n-1)\cdots 4\cdot 2}{4^{\frac{n+1}{2}}} I_{-1} & \text{if } n = \text{odd} \end{cases}$$

Note that $I_{-2} = \lim\limits_{r\to+\infty} 4\pi \int_0^r e^{-2\rho^2} d\rho$

$$= \lim_{r\to+\infty} \frac{4\pi}{\sqrt{2}} \int_0^r e^{-(\sqrt{2}\rho)^2} d(\sqrt{2}\rho)$$

$$= \lim_{r\to\infty} 2\sqrt{2}\pi \int_0^{\sqrt{2}r} e^{-t^2} dt$$

$$= 2\sqrt{2}\pi \int_0^{+\infty} e^{-t^2} dt$$

$$= \sqrt{2}\pi \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \sqrt{2}\,\pi^{3/2}$$

and $I_{-1} = \lim\limits_{r\to+\infty} 4\pi \int_0^r \rho e^{-2\rho^2} d\rho$

$$= \lim_{r\to+\infty} \pi \int_0^r e^{-2\rho^2}\, d(2\rho^2)$$

$$= \lim_{r\to+\infty} \pi \left[-e^{-2\rho^2}\right]_0^r$$

$$= \pi$$

$\therefore$

$$I_n = \begin{cases} \dfrac{(n+1)(n-1)\cdots 3\cdot 1}{4^{\frac{n}{2}+1}} \sqrt{2}\,\pi^{3/2} & \text{if } n = \text{even} \\[2ex] \dfrac{(n+1)(n-1)\cdots 4\cdot 2}{4^{\frac{n+1}{2}}}\,\pi & \text{if } n = \text{odd} \end{cases}$$

or

$$\begin{cases} I_{2k} = \dfrac{(2k+1)(2k-1)(2k-3)\cdots 1}{4^{k+1}} \cdot \sqrt{2}\cdot \pi^{3/2}, & k = 0, 1, 2, \cdots \\[2ex] I_{2k-1} = \dfrac{k!\,\pi}{2^k}, & k = 1, 2, 3, \cdots \end{cases}$$

Version 2

$$I_n = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} (x^2+y^2+z^2)^{\frac{n}{2}} e^{-\frac{1}{2}(x^2+y^2+z^2)} \, dx\,dy\,dz$$

$$= \lim_{r\to+\infty} \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho^n e^{-\frac{1}{2}\rho^2} \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \lim_{r\to+\infty} 2\pi \int_0^{\pi} \sin\phi\, d\phi \int_0^r \rho^{n+2} e^{-\frac{1}{2}\rho^2} d\rho$$

$$= \lim_{r\to+\infty} 4\pi \int_0^r \rho^{n+2} e^{-\frac{1}{2}\rho^2} d\rho$$

$$= \lim_{r\to+\infty} 4\pi \left( - \int_0^r \rho^{n+1} d e^{-\frac{1}{2}\rho^2} \right)$$

$$= \lim_{r\to+\infty} 4\pi \left[ - \rho^{n+1} e^{-\frac{1}{2}\rho^2} \Big|_0^r + \int_0^r e^{-\frac{1}{2}\rho^2} d\rho^{n+1} \right]$$

$$= \lim_{r\to+\infty} 4\pi \left[ - r^{n+1} e^{-\frac{1}{2}r^2} + (n+1) \int_0^r \rho^n e^{-\frac{1}{2}\rho^2} d\rho \right]$$

$$= (n+1) \lim_{r\to+\infty} 4\pi \int_0^r \rho^n e^{-\frac{1}{2}\rho^2} d\rho$$

$$= (n+1) I_{n-2}$$

$$\therefore \quad I_n = (n+1) I_{n-2}$$

$$= (n+1)(n-1) I_{n-4}$$

$$= \cdots$$

$$= (n+1)(n-1)\cdots(n-2k+3)\, I_{n-2k}$$

$$\Rightarrow \quad I_n = (n+1)(n-1)\cdots(n-2k+3)\cdot I_{n-2k}$$

$$= \begin{cases} (n+1)(n-1)\cdots 3\cdot 1\cdot I_{-2} & \text{if } n = \text{even} \\ (n+1)(n-1)\cdots 4\cdot 2\cdot I_{-1} & \text{if } n = \text{odd} \end{cases}$$

Note that $I_{-2} = \lim\limits_{r\to+\infty} 4\pi \int_0^r e^{-\frac{1}{2}\rho^2}\, d\rho$

$$= \lim_{r\to+\infty} 4\pi\sqrt{2}\int_0^r e^{-\left(\frac{\rho}{\sqrt{2}}\right)^2} d\left(\frac{\rho}{\sqrt{2}}\right)$$

$$= \lim_{r\to\infty} 4\sqrt{2}\,\pi \int_0^{\frac{r}{\sqrt{2}}} e^{-t^2}\, dt$$

$$= 4\sqrt{2}\,\pi \int_0^{+\infty} e^{-t^2}\, dt$$

$$= 2\sqrt{2}\,\pi \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= 2\sqrt{2}\,\pi^{3/2}$$

and $I_{-1} = \lim\limits_{r\to+\infty} 4\pi \int_0^r \rho\, e^{-\frac{1}{2}\rho^2}\, d\rho$

$$= \lim_{r\to+\infty} 4\pi \int_0^r e^{-\frac{1}{2}\rho^2}\, d\left(\frac{\rho^2}{2}\right)$$

$$= \lim_{r \to +\infty} 4\pi \left[ -e^{-\frac{\rho^2}{2}} \right]_0^r$$

$$= 4\pi$$

$\therefore$

$$I_n = \begin{cases} [(n+1)(n-1)\cdots 3\cdot 1] \cdot 2\sqrt{2}\,\pi^{3/2} & \text{if } n = \text{even} \\ [(n+1)(n-1)\cdots 4\cdot 2] \cdot 4\pi & \text{if } n = \text{odd} \end{cases}$$

or

$$\begin{cases} I_{2k} = [(2k+1)(2k-1)(2k-3)\cdots 1] \cdot 2\sqrt{2} \cdot \pi^{3/2} & , k = 0, 1, 2, \cdots \\ I_{2k-1} = k!\, 2^{k+2}\, \pi & , k = 1, 2, 3, \cdots \end{cases}$$