Remark: $\vec{\nabla}\cdot(\vec{\nabla}f) \neq 0$ in general, and it is called the Laplacian of f, and is denoted by

$$\vec{\nabla}^2 f = \vec{\nabla}\cdot(\vec{\nabla}f) = \operatorname{div}(\vec{\nabla}f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

[In graduate level, it will be denoted by $\Delta = \vec{\nabla}^2$ or $\Delta = -\vec{\nabla}^2$]

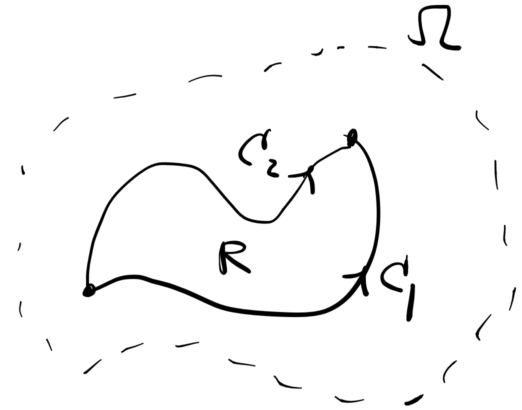
The "operator" $\vec{\nabla}^2$ is called the Laplace operator and the equation $\vec{\nabla}^2 f = 0$ is called the Laplace equation. Solutions to Laplace equation are called harmonic functions.

Pf of Thm 10 $(n=2)$

We only need to show that if Ω simply-connected (& connected) &

$$\vec{\nabla}\times\vec{F} = 0 \quad \left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$$

then $\vec{F}$ is conservative.



Case 1: C_1, C_2 have no intersection

Then "Ω is simply-connected"

$\Rightarrow$ the region R enclosed by C_1 and C_2 lies completely inside Ω.

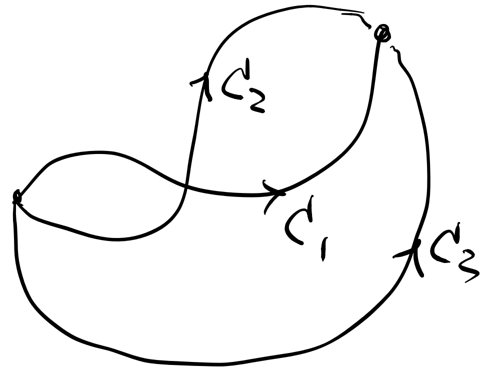
Then by Green's Thm,

$$0 = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \pm\left(\int_{C_1} - \int_{C_2}\right)(M\,dx + N\,dy)$$

$$\Rightarrow \quad \int_{C_1} M\,dx + N\,dy = \int_{C_2} M\,dx + N\,dy .$$

Case 2: C_1, C_2 intersect

Pick another curve C_3 with the same starting point and end point, and does not intersect C_1 or C_2.



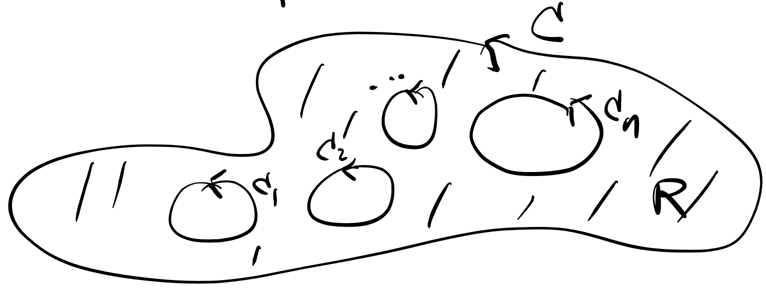
Then by case 1, $\int_{C_1} M\,dx + N\,dy = \int_{C_3} M\,dx + N\,dy$

$$= \int_{C_2} M\,dx + N\,dy$$

$\therefore \int_C \vec{F} \cdot d\vec{r}$ is independent of the path and hence conservative. ※

In order to apply Green's Thm to more general situations, we need a <u>general form of Green's Thm</u>:

Suppose that we have a simple closed curve C in $\mathbb{R}^2$



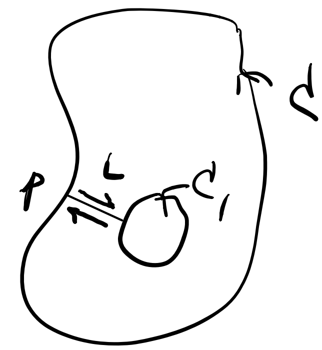
Suppose that $C_1, C_2, \cdots, C_n$ be pairwise disjoint, piecewise smooth, simple closed curves, such that $C_1, \cdots, C_n$ are enclosed by C.
(All $C, C_1, \cdots, C_n$ are anti-clockwise oriented)
Let R be the region between C and $C_1, \cdots, C_n$.
Suppose that $\vec{F} = M\vec{i} + N\vec{j}$ is defined on some open set containing R, and is C^1. Then we have

$$\boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M dx + N dy - \sum_{i=1}^{n} \oint_{C_i} M dx + N dy}$$

(This is the tangential form. The normal form is similar)

<u>Sketch of Proof</u>

For simplicity, only one C_1 inside C.
We connect C & C_1 by an "arc" L
and consider the "simple" closed curve



(starting from p):

$$C^* = C + L - C_1 - L$$

Then the region R enclosed between C & C_1 is the region enclosed by C^* except the arc L.

Hence $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_{R \setminus L} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$\overset{\text{Green's}}{=} \oint_{C^*} M\,dx + N\,dy$$

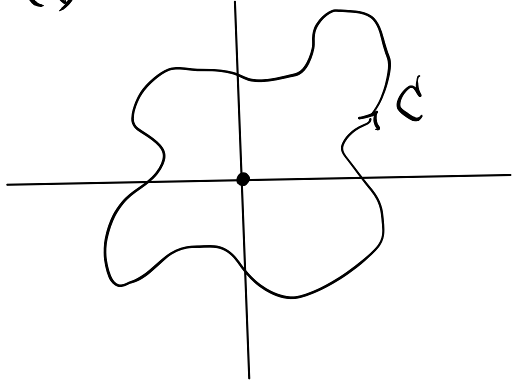
$$= \oint_C M\,dx + N\,dy - \oint_{C_1} M\,dx + N\,dy$$ #

eg49: $\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$ on $\mathbb{R}^2 \setminus \{(0,0)\} = \Omega$

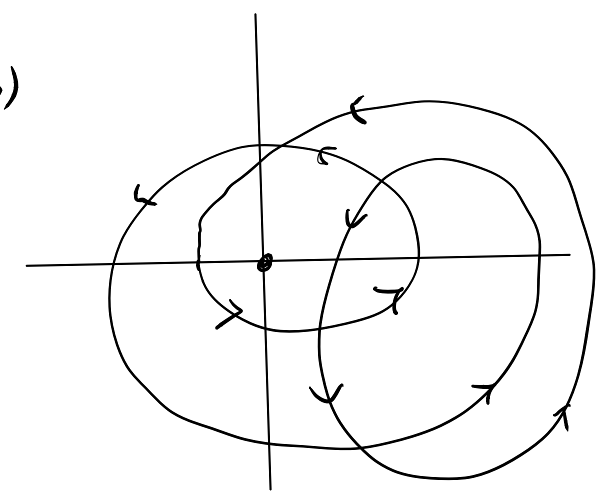
we've calculated $\oint_{C_1} \vec{F} \cdot d\vec{r} = 2\pi$ for $C_1 : x^2+y^2 = 1$ (anti-clockwise)

How about

(a)



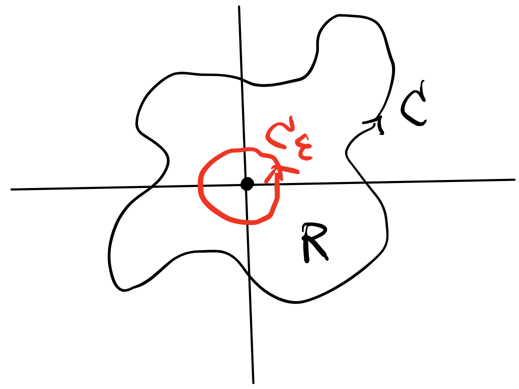
(b)



Soln (a) Recall that $\vec{\nabla}\times\vec{F}=0$

(Green's Thm doesn't apply to get $\oint_C \vec{F}\cdot d\vec{r}=0$, since C encloses the origin $(0,0)$, where $\vec{F}$ is not defined.)

Choose $\varepsilon>0$ small enough, such that the circle C_ε of radius ε centered at $(0,0)$ is completely enclosed by C



$\vec{F}$ is smooth in the region R between C and C_ε.

Hence the general form of Green's Thm applied:

$$0=\iint_R (\vec{\nabla}\times\vec{F})\cdot\hat{k}\,dA=\oint_C \vec{F}\cdot d\vec{r}-\oint_{C_\varepsilon}\vec{F}\cdot d\vec{r}$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r}=\oint_{C_\varepsilon}\vec{F}\cdot d\vec{r}$$

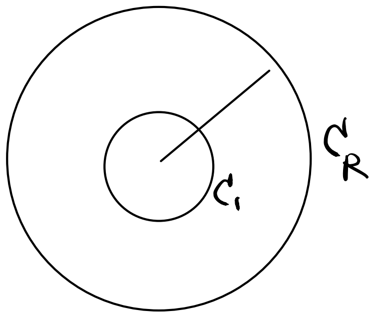
$$=\oint_{C_\varepsilon}\frac{-y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy$$

Parametrize C_ε by $\begin{cases} x=\varepsilon\cos\theta \\ y=\varepsilon\sin\theta\end{cases}$, $0\le\theta\le2\pi$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r}=\int_0^{2\pi}\left[-\frac{\varepsilon\sin\theta}{\varepsilon^2}(-\varepsilon\sin\theta)+\frac{\varepsilon\cos\theta}{\varepsilon^2}(\varepsilon\cos\theta)\right]d\theta$$

$$=\int_0^{2\pi}d\theta=2\pi$$

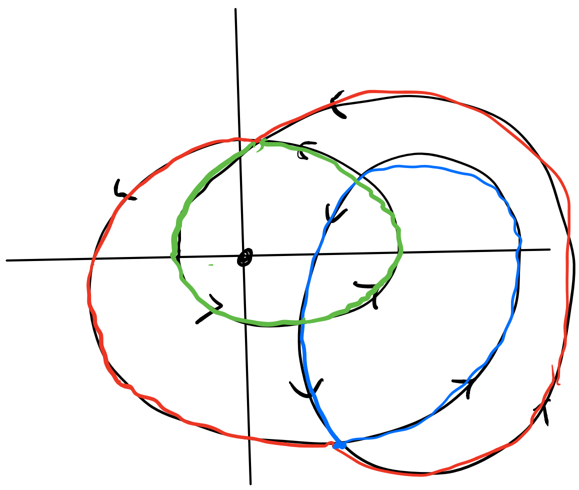
(In fact, we've proved that $\oint_{C_R} \vec{F}\cdot d\vec{r} = 2\pi$, $\forall$ any radius $R>0$, which can be seen by consider the domain between C_1 & C_R



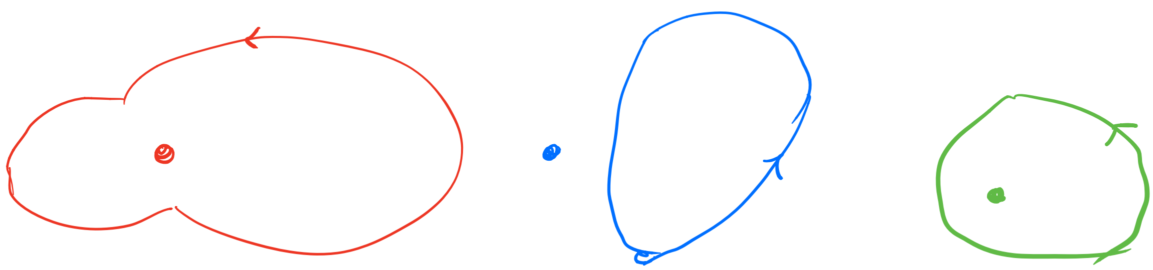
Green's thm

$$\Rightarrow \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_R} \vec{F}\cdot d\vec{r}$$)

(b)



Recompose the curve into



2π $\qquad$ 0 $\qquad$ 2π

Hence $\oint_C \vec{F}\cdot d\vec{r} = 2\pi + 0 + 2\pi = 4\pi$