Triple Integrals

Def 5 Let $f(x,y,z)$ be a function defined on a (closed and bounded) rectangular box

$$B = [a,b] \times [c,d] \times [r,s]$$

Then the triple integral of f over the box B is

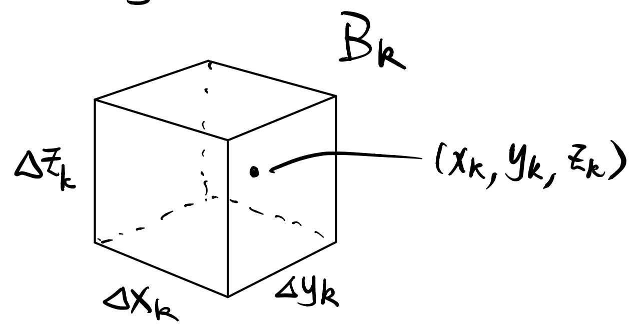
$$\iiint_B f(x,y,z)\,dV = \lim_{\|P\| \to 0} \sum_k f(x_k, y_k, z_k)\,\Delta V_k$$

if this exists.

where (i) $P = P_1 \times P_2 \times P_3$ is a subdivision of B into sub-rectangular boxes by partitions P_1, P_2, P_3 of $[a,b]$, $[c,d]$, & $[r,s]$ respectively. And

$$\|P\| = \max\left(\|P_1\|, \|P_2\|, \|P_2\|\right)$$

(ii) (x_k, y_k, z_k) is an arbitrary point in a sub-rectangular box B_k



(iii) $\Delta V_k = \mathrm{Vol}(B_k) = \Delta x_k \, \Delta y_k \, \Delta z_k$

Thm 4 (Fubini's Theorem for Triple Integrals (1st form))

If $f(x,y,z)$ is continuous (in fact, "absolutely" integrable is sufficient) on $B=[a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Note: Interchanging the order of the coordinates, we also have

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$$

$$= \cdots \text{ in any order of } dx, dy, dz.$$

Def 6 (Triple integral over a general region $D\subset\mathbb{R}^3$)

Let $f(x,y,z)$ be a function on a closed and bounded region $D\subset\mathbb{R}^3$. Then

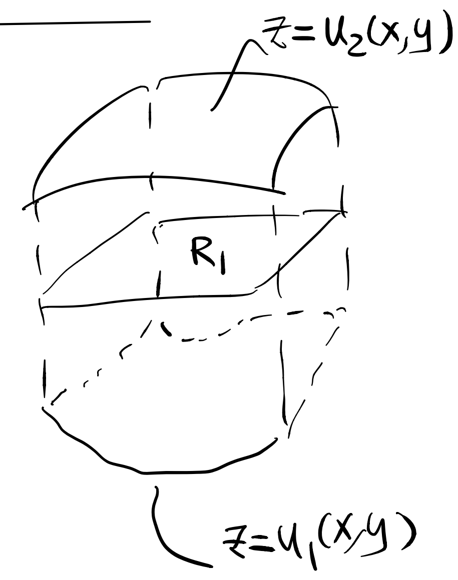
$$\iiint_D f(x,y,z)\,dV \overset{\text{def}}{=} \iiint_B F(x,y,z)\,dV$$

where B is a closed and bounded rectangular box containing D, and

$$F(x,y,z) = \begin{cases} f(x,y,z), & \text{if } (x,y,z)\in D \\ 0, & \text{if } (x,y,z)\in B\setminus D \end{cases}$$

Note: As in double integral, this definition is well-defined.

Special types of closed and bounded region $D \subset \mathbb{R}^3$



(1) $D = \{(x,y,z) : (x,y) \in R_1,\ u_1(x,y) \le z \le u_2(x,y)\}$

$(u_1(x,y) \le u_2(x,y),\ u_1 \not\equiv u_2)$

(2) $D = \{(x,y,z) : (x,z) \in R_2,\ v_1(x,z) \le y \le v_2(x,z)\}$

$(v_1 \le v_2,\ v_1 \not\equiv v_2)$

(3) $D = \{(x,y,z) : (y,z) \in R_3,\ w_1(y,z) \le x \le w_2(y,z)\}$

$(w_1 \le w_2,\ w_1 \not\equiv w_2)$

where R_i, $i=1,2,3$ are closed and bounded plane regions and u_1, u_2; v_1, v_2; w_1, w_2 are continuous wrt the corresponding variables.

Thm 5 (Fubini's Thm for triple integrals (Strong form))

Let $f(x,y,z)$ be a continuous (absolutely integrable) function on D

If D is of type (1) as above, then

$$\iiint_D f(x,y,z)\,dV = \iint_{R_1} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dx\,dy$$

Similarly for types (2) and (3).

Note: Particularly, we have (using Fubini's for double integrals)

if $D = \left\{ (x,y,z) : \begin{array}{l} a \le x \le b,\ g_1(x) \le y \le g_2(x) \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}$

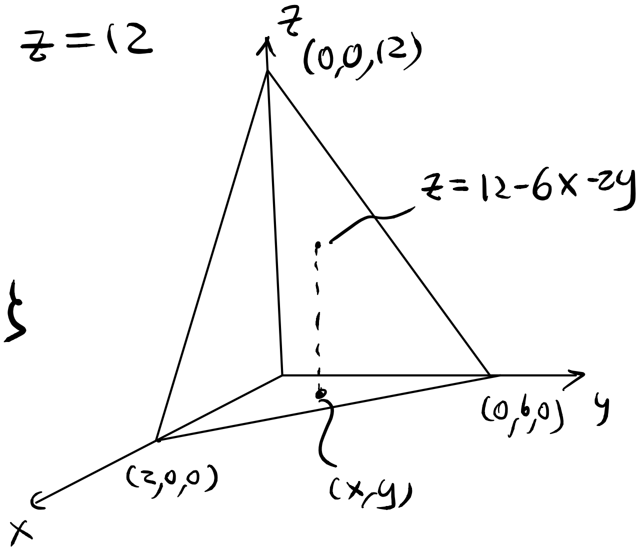
(i.e. R_1 is of type (1) as in double integrals), then

$$\boxed{\iiint_D f(x,y,z) = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

Similarly for other types.

> Prop 6: The propositions 1-4 for double integrals also hold for triple integrals over closed and bounded region in $\mathbb{R}^3$.

eg 17 Volume of the bounded region D in the 1st octant enclosed by the plane $6x+2y+z=12$



Soln: D is of special type

$= \{ (x,y) \in R_1 : 0 \le z \le 12-6x-2y \}$

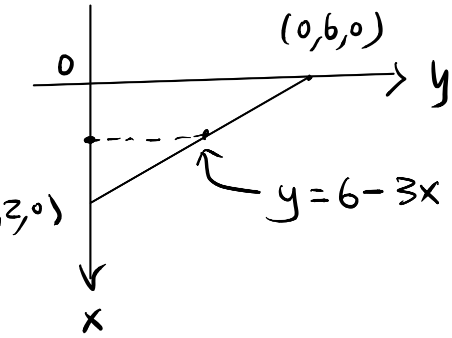
$= \left\{ \begin{array}{l} 0 \le x \le 2,\ 0 \le y \le 6-3x, \\ 0 \le z \le 12-6x-2y \end{array} \right\}$

$\Rightarrow$

$$\mathrm{Vol}(D) = \iiint_D 1 \cdot dV$$

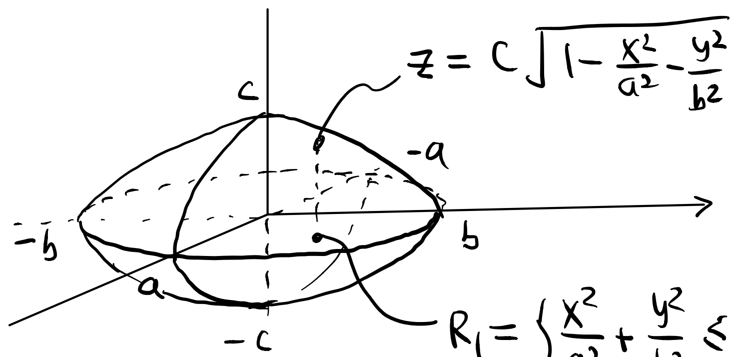
$$= \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} 1 \cdot dz\,dy\,dx$$

$$= \cdots = 24 \quad (\text{check!})$$



eg18 : Volume of Ellipsoid

$$D = \left\{ (x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\} \quad (a,b,c>0)$$



$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$R_1 = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\} \quad (z=0)$$

$$= \left\{ 0 \le x \le a, \; 0 \le y \le b\sqrt{1 - \frac{x^2}{a^2}} \right\}$$

By symmetry, we can consider the 1st octant only

$\Rightarrow$ $\mathrm{Vol}(D) = 8 \cdot$ volume of D in the 1st octant

$$= 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\,dx$$

$$= \cdots = \frac{4\pi abc}{3} \quad \text{(optimal exercise)}$$

In fact, we have, similarly

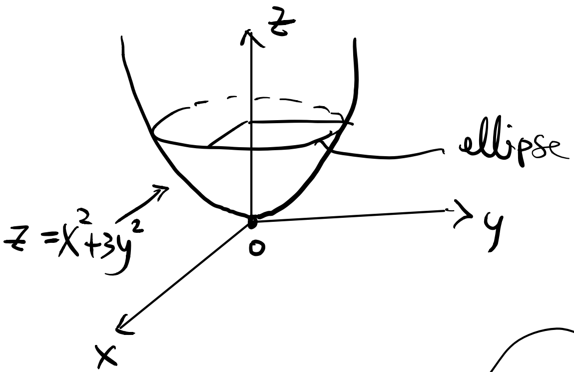
$$\mathrm{Vol}(D) = 8 \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \int_0^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}} dx\,dy\,dz$$

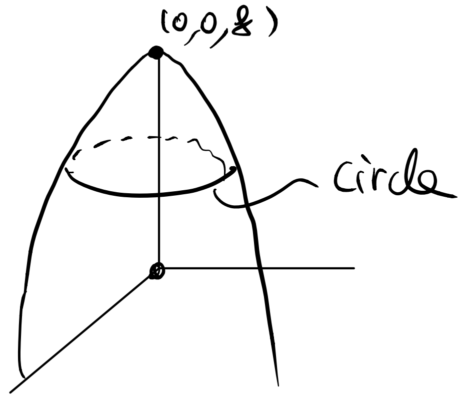
and etc. (optional exercise)

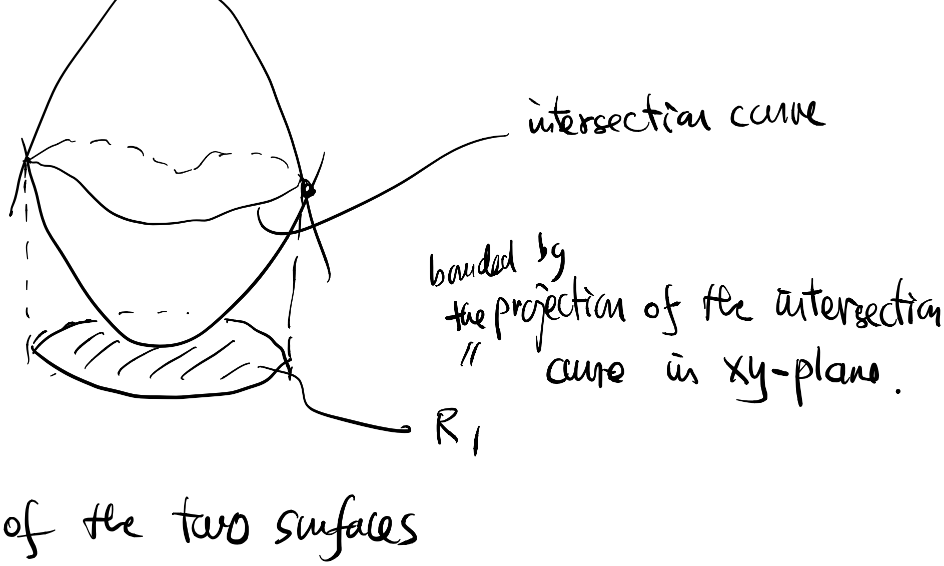
[Note: better way to do this is by change of variables formula (later in the course).]

<u>eg 19</u>: Find the volume of D enclosed by

$$z = x^2 + 3y^2 \quad \text{and} \quad z = 8 - x^2 - y^2$$







At the <u>intersection</u> of the two surfaces

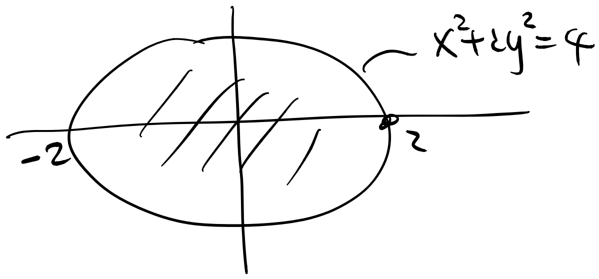
$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

$$\Rightarrow \qquad x^2+3y^2 = 8-x^2-y^2$$

is the projection (in xy-plane) of the intersection curve

which is $\quad x^2+2y^2=4 \quad$ (a ellipse)

So R_1 is



$$\Rightarrow D = \left\{ (x,y) \in R_1 = \{ x^2+2y^2 \le 4 \},\ x^2+3y^2 \le z \le 8-x^2-y^2 \right\}$$

$$= \left\{ -2 \le x \le 2,\ -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}},\ x^2+3y^2 \le z \le 8-x^2-y^2 \right\}$$

$$\Rightarrow \mathrm{Vol}(D) = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-4y^2)\,dy\,dx$$

$$= \int_{-2}^{2} \frac{4\sqrt{2}}{3} (4-x^2)^{3/2}\,dx \qquad \text{(check!)}$$

$$= 8\pi\sqrt{2} \qquad \text{(check!)}$$

[For those interested in the intersection (space) curve (in parametric form)

$x = 2\cos t,\quad y = \sqrt{2}\sin t,\quad z = 4 + 2\sin^2 t \qquad (0 \le t \le 2\pi)$]

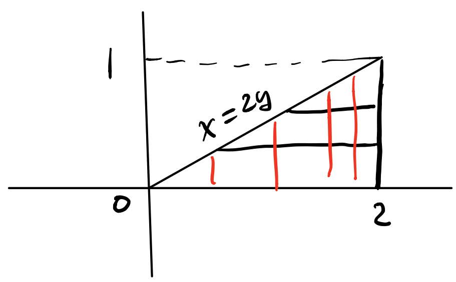
eg20 Evaluate $\displaystyle\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz$

$$= \int_0^4 \frac{2}{\sqrt{z}} \Big[\underbrace{\int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy}_{} \Big] dz$$

this double integral doesn't depend on z

$$= \left(\int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy \right) \left(\int_0^4 \frac{2}{\sqrt{z}}\, dz \right)$$

think of this as double integral over the region



By Fubini's

$$= \left(\int_0^2 \int_0^{\frac{x}{2}} \cos(x^2)\, dy\, dx \right) \left(\int_0^4 \frac{2}{\sqrt{z}}\, dz \right)$$

$$= \int_0^2 \Big[\cos(x^2) \Big(\int_0^{\frac{x}{2}} dy \Big) \Big] dx \left(\int_0^4 \frac{3}{\sqrt{z}}\, dz \right)$$

$$= \left(\int_0^2 \frac{x}{2} \cos(x^2)\, dx \right) \left(\int_0^4 \frac{2}{\sqrt{z}}\, dz \right)$$

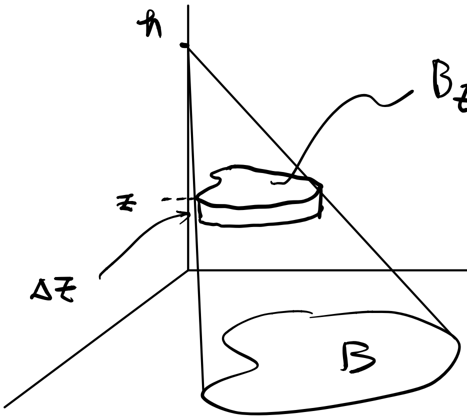
Then the integration can be easily evaluated!

$$= 2\sin 4 \quad \text{(check!)}$$

✳

eg21 Let B (base) be a "nice" subset of $\mathbb{R}^2$

Let D = cone in $\mathbb{R}^3$ with base B on xy-plane and vertex $(0, 0, h)$ $(h>0)$

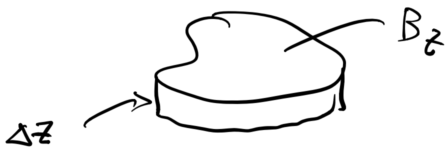


$B_z = D \cap \{\text{horizontal plane at height } z\}$

(B_z is "similar" to B)

How to find the volume of D?

Answer: by concept of Riemann sum and this figure



$\text{Vol} \sim \text{Area}(B_z) \cdot \Delta z$

$$\Rightarrow \quad \text{Vol}(D) = \int_0^h \text{Area}(B_z)\, dz$$

By similarity: ratio of height: $\dfrac{h-z}{h} = 1 - \dfrac{z}{h}$

$$\Rightarrow \text{ratio of area}: \quad \frac{\text{Area}(B_z)}{\text{Area}(B)} = \left(1 - \frac{z}{h}\right)^2$$

$$\Rightarrow \quad \text{Vol}(D) = \int_0^h \left(1 - \frac{z}{h}\right)^2 \text{Area}(B)\, dz$$

$$= \int_0^h \left(1 - \frac{z}{h}\right)^2 dz \cdot \text{Area}(B)$$

$$= \frac{h}{3}\,\text{Area}(B) \quad \text{※} \quad (\text{check!})$$