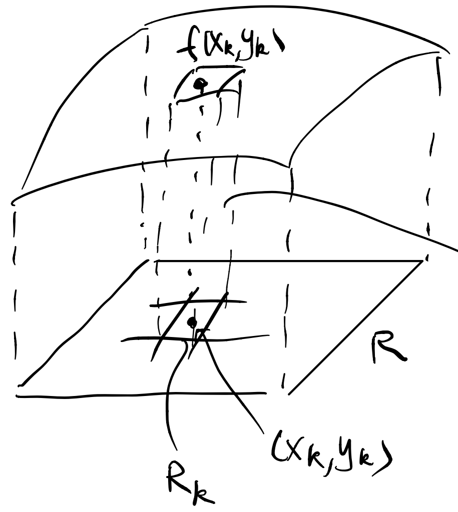
Remark : Same as 1-variable, the double integral of f, $f \geq 0$, over R can be interpreted as volume under the graph of f (over R)

$f(x,y) \geq 0$

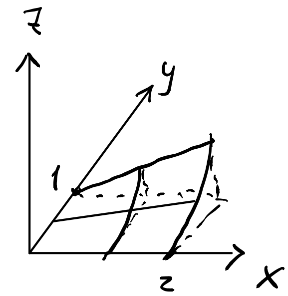


Volume under the graph of f over sub-rectangle R_k

$\sim f(x_k, y_k) \Delta A_k$

eg2 : $R = [0,2] \times [0,1]$, $f(x,y) = xy^2$

(Using definition) Find $\iint_R xy^2 \, dx dy$.



Soln: Using the uniform partitions :

$$P_1 = \{0, \tfrac{2}{n}, \tfrac{4}{n}, \cdots, 2\} \text{ of } [0,2]$$

$$P_2 = \{0, \tfrac{1}{n}, \tfrac{2}{n}, \cdots, 1\} \text{ of } [0,1]$$

$\Rightarrow$ a particular sub-rectangle is

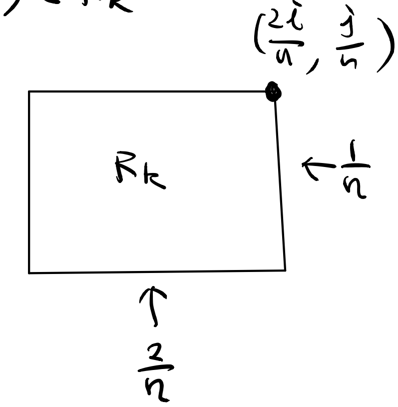
$$R_k = \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$$

for some $i = 1, \cdots, n$; $j = 1, \cdots, n$

(So R_k should be better denoted by R_{ij})

(Assume it is integrable)

One may choose the point $(x_k, y_k) = \left(\frac{2i}{n}, \frac{j}{n}\right) \in R_k$ and consider the Riemann sum



$$\sum_k f(x_k, y_k)\,\Delta A_k$$

$$= \sum_{i,j=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{j}{n}\right)^2 \cdot \frac{2}{n} \cdot \frac{1}{n}$$

$$= \frac{4}{n^5} \sum_{i,j=1}^{n} i j^2$$

$$= \frac{4}{n^5} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} i j^2 \right)$$

$$= \frac{4}{n^5} \sum_{i=1}^{n} \left[i \left(\sum_{j=1}^{n} j^2 \right) \right]$$

$$= \frac{4}{n^5} \left(\sum_{i=1}^{n} i \right) \left(\sum_{j=1}^{n} j^2 \right)$$

$$= \frac{4}{n^5} \cdot \frac{n(n+1)}{2} \cdot \frac{n(n+1)(2n+1)}{6} \longrightarrow \frac{4 \cdot 2}{2 \cdot 6} = \frac{2}{3} \quad \text{as } n \to \infty$$

$$\therefore \iint_{[0,2]\times[0,1]} x y^2 \, dx\, dy = \frac{2}{3} \qquad \#$$

Very tedius calculation

Hence we need the following Theorem :

Thm 1 (Fubini's Theorem (1st form))

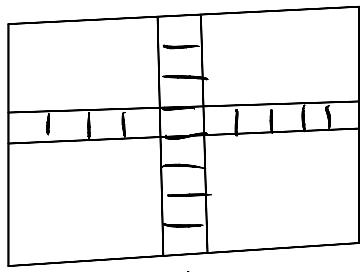
If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\, dA = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx .$$

The last 2 integrals above are called <u>iterated integrals</u>.

(Pf: Omitted)

Ideas:



sum horizontal first & taking limit →

$$\int_c^d \left[\int_a^b f(x,y)\,dx\right]dy$$

("$\sum_i$" first in eg2)

← sum vertically first & taking limit

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right]dx$$

("$\sum_j$" first in eg2)

<u>eg3</u>: Using Fubini to calculate $\iint_R xy^2\,dxdy$ where $R=[0,2]\times[0,1]$

<u>Soln</u>: By Fubini

$$\iint_R xy^2\,dA = \int_0^2\left[\int_0^1 xy^2\,dy\right]dx$$

$$= \int_0^2\left(x\int_0^1 y^2\,dy\right)dx$$

$$= \int_0^2 \frac{x}{3}\,dx = \frac{2}{3}.$$

or $$\iint_R xy^2\,dA = \int_0^1\left[\int_0^2 xy^2\,dx\right]dy$$

$$= \int_0^1\left[y^2\int_0^2 x\,dx\right]dy$$

$$= \int_0^1 2y^2\,dy = \frac{2}{3}$$

Much easier than using Riemann sum! ※

<u>eg4</u>: Some times the "order" of the iterated integrals is important in practical calculations! Find $\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA$.

Soln: $\displaystyle\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^{\pi}\left[\int_0^1 x\sin(xy)\,dx\right]dy$

$$= \int_0^{\pi}\left[-\frac{\cos y}{y} + \frac{\sin y}{y^2}\right]dy \quad \text{(integration-by-parts)}$$

Not easy to integrate!

On the other hand, in different order

$$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^1\left[\int_0^{\pi} x\sin(xy)\,dy\right]dx$$

$$= \int_0^1 \left[-\cos xy\right]_{y=0}^{y=\pi} dx$$

$$= \int_0^1 (-\cos\pi x + 1)\,dx$$

$$= 1 \quad \text{(easy!)}$$

#

Caution: Not all functions are integrable over a (closed) rectangle.

Remark:
- To show "integrable", needs to show that for all partitions and for all points (x_k, y_k) in the subrectangles, the Riemann sum $S(f,P) \to$ the same number (as $\|P\| \to 0$)
- To disprove "integrable", needs to find, for examples,
  - (i) some P with some choose of (x_k, y_k) such that $\lim_{\|P\|\to 0} S(f,P)$ doesn't exist.
  - (ii) some P with different (x_k, y_k) & (x_k', y_k') such that

$$\underset{\text{with } (x_k,y_k)}{S(f,P)} \to a \neq b \leftarrow \underset{\text{with } (x_k',y_k')}{S'(f,P)}$$

eg5: Let $R=[0,1]\times[0,1]$

$$f(x,y)=\begin{cases}0, & \text{if both } x \text{ \& } y \text{ are rational}\\ 1, & \text{otherwise}\end{cases}$$

Then f is not integrable over R.

Soln: $\forall$ subdivision (partition) P of $R = R_1\cup R_2\cup\cdots\cup R_n$

One can find points $(x_k,y_k)\in R_k$, for any k, such that both x_k, y_k are rational.

The corresponding Riemann sum equals

$$S_n(f,P)=\sum_{k=1}^{n} f(x_k,y_k)\Delta A_k=\sum_{k=1}^{n} 0\cdot\Delta A_k = 0 \to 0 \text{ as } \|P\|\to 0.$$

On the other hand, we can also find $(x_k',y_k')\in R_k$ such that at least one of x_k', y_k' is irrational.

The corresponding Riemann sum equals

$$S_n'(f,P)=\sum_{k=1}^{n} f(x_k',y_k')\Delta A_k=\sum_{k=1}^{n} 1\,\Delta A_k$$

$$= \text{area of } R = 1 \quad \text{as } \|P\|\to 0$$

Since $S_n(f,P)\to 0 \neq 1 \leftarrow S_n'(f,P)$,

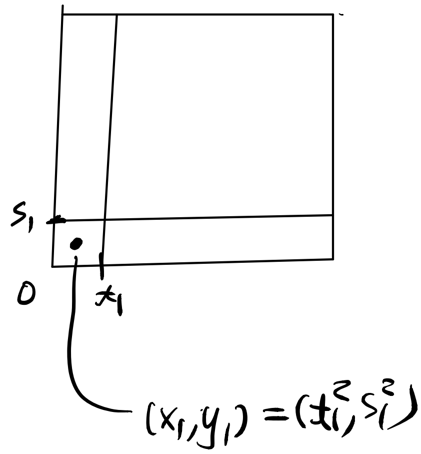
f is not integrable. ※

eg6: Let $R=[0,1]\times[0,1]$ $\quad f(x,y)=\begin{cases}\frac{1}{xy}, & \text{if } x\neq 0 \text{ \& } y\neq 0\\ 0, & \text{if } x=0, \text{ or } y=0\end{cases}$

Then f is not integrable over R. $\quad (f(x,y)\geq 0)$

Pf: In any partition P of R, there is a sub-rectangle

$R_1 = [0, t_1] \times [0, s_1]$.



Choose

$(x_1, y_1) = (t_1^2, s_1^2) \in R_1$

(since $0 < t_1^2 < t_1 < 1$, $0 < s_1^2 < s_1 < 1$)

Then Riemann sum

$$S(f, P) = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

$$= f(x_1, y_1)\Delta A_1 + \sum_{k=2}^{n} f(x_k, y_k)\Delta A_k$$

$$\geq f(t_1^2, s_1^2)\cdot t_1 s_1$$

$$= \frac{1}{t_1^2 s_1^2}\cdot t_1 s_1 = \frac{1}{t_1 s_1}$$

Since $0 < t_1, s_1 \leq \|P\| \to 0$, $t_1, s_1 \to 0$

Hence $S(f, P) \geq \dfrac{1}{t_1 s_1} \to \infty$ as $\|P\| \to 0$

$\therefore$ limit doesn't exist!

Hence f is <u>not</u> integrable. ※

By egs. 5 & 6, we see that "condition(s)" is need to ensure integrability of a function over (closed) rectangle.

Prop 1: Let $R=[a,b]\times[c,d]$ be a closed rectangle, and $f(x,y)$ be an integrable function over R, then f is bounded on R

(i.e. $\exists M>0$ s.t. " $|f(x,y)|\le M$, $\forall (x,y)\in R$ ")

Pf: Omitted (eg6 above gives an idea of proof.)

Remark: From eg5, "boundedness" is necessary, but not sufficient, for integrability.

Prop 2: Let $R=[a,b]\times[c,d]$ be a closed rectangle, and $f(x,y)$ be a continuous function on R, then f is integrable on R.

Pf: Omitted (See proof in 1-variable case in MATH 2060 for an ideal of proof.)

Remarks: (i) Note that a continuous function on closed rectangle is always bounded (Props. 1 & 2 are consistent) (MATH2050 for 1-variable situation).

(ii) "continuity" is sufficient, but not necessary. (on closed rectangles). In fact, Prop 2 can be generalized to a bounded function on a closed rectangle with a "small" set of discontinuity. The precise concept is "measure zero set" (MATH 4050 Real Analysis)

For us, we have

Prop 2' For function over closed rectangle.

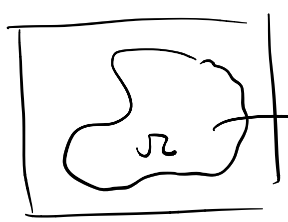
(a) bounded + "continuous except finitely many points"

$\Rightarrow$ integrable

(b) bounded + "continuous except finitely many differentiable curves"

$\Rightarrow$ integrable.

eg



$$f(x,y) = \begin{cases} \text{continuous on } \Omega \text{ (and bounded)} \\ 0 \quad \text{otherwise} \end{cases}$$

Furthermore, we have

Prop 3: Let $R = [a,b] \times [c,d]$ be a closed rectangle, $f(x,y)$ and $g(x,y)$ be functions on R, and $k \in \mathbb{R}$ is a constant.

(1) If f & g are integrable over R, then $f \pm g$ and kf are integrable over R.

(2) In the case of (1), we have

$$\iint_R [f \pm g](x,y)\, dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA$$

and

$$\iint_R k f(x,y)\, dA = k \iint_R f(x,y)\, dA$$

Pf: Omitted (Obvious from the concept of Riemann sum)

<u>Remark</u>: This Prop3 implies that the set of integrable functions over R forms a "vector space over $\mathbb{R}$", and "(double) integral" is <u>linear</u> (when the rectangle R is fixed).