MATH 2020 Advanced Calculus II

Solutions to HW8

Due date: 8 Nov

Section 16.3

8. A potential function for $\mathbf{F}$ is given by

$$f = xy + yz + zx.$$

13.

$$2xdx + 2ydy + 2zdz = d(x^2 + y^2 + z^2)$$

$$\begin{aligned}\text{integral} &= \left[x^2 + y^2 + z^2\right]_{(0,0,0)}^{(2,3,-6)} \\ &= 49\end{aligned}$$

18. Notice that

$$2\cos y dx + \left(\frac{1}{y} - 2x \sin y\right) dy + \frac{1}{z} dz = d(2x\cos y + \ln y + \ln z).$$

$$\begin{aligned}\text{integral} &= [2x\cos y + \ln y + \ln z]_{(0,2,1)}^{(1,\frac{\pi}{2},2)} \\ &= \left(\ln\frac{\pi}{2} + \ln 2\right) - \ln 2 \\ &= \ln\frac{\pi}{2}\end{aligned}$$

20. Notice that

$$(2x\ln y - yz)dx + \left(\frac{x^2}{y} - xz\right) dy - xydz = d(x^2\ln y - xyz).$$

$$\begin{aligned}\text{integral} &= \left[x^2\ln y - xyz\right]_{(1,2,1)}^{(2,1,1)} \\ &= (-2) - (\ln 2 - 2) \\ &= -\ln 2\end{aligned}$$

29. Notice that

$$\mathbf{F} = \nabla\left(\frac{x^3}{3} + xy + \frac{y^3}{3} + ze^z - e^z\right).$$

The integrals for parts (a),(b),(c) are all equal to

$$\left[\frac{x^3}{3} + xy + \frac{y^3}{3} + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1.$$

Section 16.4

6. Let R be the region bounded by $x = 0, x = 1, y = 0, y = 1$.

$$\begin{aligned}
\text{circulation} &= \oint_C \mathbf{F} \cdot \mathbf{T} ds \\
&= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \\
&= \int_0^1 \int_0^1 (1 - 4) dx dy \\
&= -3
\end{aligned}$$

$$\begin{aligned}
\text{flux} &= \oint_C \mathbf{F} \cdot \mathbf{n} ds \\
&= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \\
&= \int_0^1 \int_0^1 (2x + 2y) dx dy \\
&= 2
\end{aligned}$$

13. Let R be the region bounded by $r^2 = \cos 2\theta,\ \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

$$\begin{aligned}
\text{circulation} &= \oint_C \mathbf{F} \cdot \mathbf{T} ds \\
&= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \\
&= \iint_R (1 + e^x \cos y - e^x \cos y) dx dy \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} r dr d\theta \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{2} \cos 2\theta \right) d\theta \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\text{flux} &= \oint_C \mathbf{F} \cdot \mathbf{n} ds \\
&= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \\
&= \iint_R [1 + e^x \sin y + (-e^x \sin y)] dx dy \\
&= \frac{1}{2}
\end{aligned}$$

22. Let R be the region bounded below by the axis and bounded above by $y = \sin x,\ x \in [0, \pi]$.

$$\begin{aligned} & \oint_C 3y dx + 2x dy \\ = & \iint_R \left[\frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(3y) \right] dx dy \\ = & \iint_R (-1) dx dy \\ = & -\int_0^{\pi} \int_0^{\sin x} dy dx \\ = & -\int_0^{\pi} \sin x dx \\ = & -2 \end{aligned}$$

30. Let C be the boundary (oriented counter-clockwise) of a square R in the plane.

$$\begin{aligned} & \oint_C xy^2 dx + (x^2 y + 2x) dy \\ = & \iint_R \left[\frac{\partial}{\partial x}(x^2 y + 2x) - \frac{\partial}{\partial y}(xy^2) \right] dx dy \\ = & \iint_R (2xy + 2 - 2xy) dx dy \\ = & 2 \times \text{Area}(R) \end{aligned}$$

34. Let R be the region bounded by C.

$$\begin{aligned} & -\oint_C y dx \\ = & -\iint_R \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(y) \right] dx dy \\ = & \iint_R dx dy \\ = & \int_a^b \int_0^{f(x)} dy dx \\ = & \int_a^b f(x) dx \end{aligned}$$