MATH 2020 Advanced Calculus II

Solutions to HW7

Due date: 1 Nov

Section 16.1

12. We have $\mathbf{r}'(t) = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3\mathbf{k}$ and $|\mathbf{r}'(t)| = 5$.

$$\begin{aligned}\int_C \sqrt{x^2+y^2}ds &= \int_{-2\pi}^{2\pi} \sqrt{(4\cos t)^2 + (4\sin t)^2}\cdot 5dt \\ &= 20\int_{-2\pi}^{2\pi} dt \\ &= 80\pi\end{aligned}$$

13. Parametrize the segment by $\mathbf{r}(t) = (1-t, 2-3t, 3-2t),\ t\in[0,1]$. Then $\mathbf{r}'(t) = (-1,-3,-2)$ and $|\mathbf{r}'(t)| = \sqrt{14}$.

$$\begin{aligned}\text{integral} &= \int_0^1 [(1-t)+(2-3t)+(3-2t)]\cdot\sqrt{14}dt \\ &= \int_0^1 (6-6t)\sqrt{14}dt \\ &= 3\sqrt{14}\end{aligned}$$

18. We have $\mathbf{r}'(t) = (-a\sin t)\mathbf{j} + (a\cos t)\mathbf{k}$ and $|\mathbf{r}'(t)| = a$.

$$\begin{aligned}\text{integral} &= \int_0^{2\pi} -\sqrt{0^2+(a\sin t)^2}\cdot adt \\ &= -a^2\int_0^{2\pi} |\sin t|dt \\ &= -2a^2\int_0^{\pi} \sin t\ dt \\ &= -4a^2\end{aligned}$$

24. We have $\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t^3\mathbf{j}$ and $|\mathbf{r}'(t)| = \sqrt{(3t^2)^2+(4t^3)^2} = t^2\sqrt{9+16t^2}$.

$$\begin{aligned}\text{integral} &= \int_{\frac{1}{2}}^1 \frac{\sqrt{t^4}}{t^3}\cdot t^2\sqrt{9+16t^2}dt \\ &= \int_{\frac{1}{2}}^1 t\sqrt{9+16t^2}dt \\ &= \left[\frac{\sqrt{9+16t^2}^3}{48}\right]_{\frac{1}{2}}^1 \\ &= \frac{5^3-\sqrt{13}^3}{48} \\ &= \frac{125-13\sqrt{13}}{48}\end{aligned}$$

Section 16.2

8. (a) We have $\mathbf{r}'(t) = (1, 1, 1)$ and $F \cdot \mathbf{r}' = \dfrac{1}{t^2+1}$.

$$\begin{aligned}\text{integral} &= \int_0^1 \frac{1}{t^2+1}dt \\ &= \frac{\pi}{4}\end{aligned}$$

(b) We have $\mathbf{r}'(t) = (1, 2t, 4t^3)$ and $F \cdot \mathbf{r}' = \dfrac{2t}{t^2+1}$.

$$\begin{aligned}\text{integral} &= \int_0^1 \frac{2t}{t^2+1}dt \\ &= \ln 2\end{aligned}$$

(c) Let C_3 and C_4 be parametrized by

$$\mathbf{r}_1(t) = (t, t, 0),\ t \in [0, 1] \quad \text{and} \quad \mathbf{r}_2(t) = (1, 1, t),\ t \in [0, 1]$$

respectively. We have $\mathbf{r}_1'(t) = (1, 1, 0)$, $\mathbf{r}_2'(t) = (0, 0, 1)$ and $F \cdot \mathbf{r}_1' = \dfrac{1}{t^2+1}$, $F \cdot \mathbf{r}_2' = 0$.

$$\begin{aligned}\text{integral} &= \int_0^1 \frac{1}{t^2+1}dt + 0 \\ &= \frac{\pi}{4}\end{aligned}$$

16. Notice that $dx = 0$ for the vertical segment from $(0, 3)$ to $(0, 0)$. As for the other two:

$$\mathbf{r}_1(t) = (t, 3t),\ t \in [0, 1] \quad \text{and} \quad \mathbf{r}_2(t) = (1-t, 3),\ t \in [0, 1],$$

we have $dx = dt$ and $dx = -dt$ respectively.

$$\begin{aligned}\text{integral} &= \int_0^1 \sqrt{t+3t} \cdot dt + \int_0^1 \sqrt{1-t+3} \cdot (-dt) \\ &= 2\left[\frac{2\sqrt{t}^3}{3}\right]_0^1 - \left[-\frac{2\sqrt{4-t}^3}{3}\right]_0^1 \\ &= \frac{4}{3} + \frac{2}{3}(3\sqrt{3} - 8) \\ &= 2\sqrt{3} - 4\end{aligned}$$

28. Let $\mathbf{r}(t) = (2\cos t, 2\sin t),\ t \in [0, 2\pi]$ parametrize the circle. Then

$$\mathbf{r}'(t) = (-2\sin t, 2\cos t).$$

We have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = 2(x+y)\mathbf{i} + 2(x+y)\mathbf{j}.$$

$$
\begin{aligned}
\text{Work done} &= \int_C \nabla f \cdot ds \\
&= \int_0^{2\pi} 2(4\cos t + 4\sin t, 4\cos t + 4\sin t) \cdot (-2\sin t, 2\cos t)dt \\
&= 16\int_0^{2\pi} (\cos t + \sin t)(\cos t - \sin t)dt \\
&= 16\int_0^{2\pi} (\cos^2 t - \sin^2 t)dt \\
&= 0
\end{aligned}
$$

35. (a) Parametrize the curve by $\mathbf{r}(t) = (\cos t, \sin t),\ t \in [0, \pi]$.
Then $\mathbf{r}'(t) = (-\sin t, \cos t)$.

$$
\begin{aligned}
\int_C F \cdot ds &= \int_0^{\pi} (\cos t + \sin t, -\cos^2 t - \sin^2 t) \cdot (-\sin t, \cos t)dt \\
&= \int_0^{\pi} (-\sin t\cos t - \sin^2 t - \cos t)dt \\
&= -\frac{\pi}{2}
\end{aligned}
$$

(b) Parametrize the curve by $\mathbf{r}(t) = (1 - 2t, 0),\ t \in [0, 1]$. Then $\mathbf{r}'(t) = (-2, 0)$.

$$
\begin{aligned}
\int_C F \cdot ds &= \int_0^{1} (1 - 2t, -(1 - 2t)^2) \cdot (-2, 0)dt \\
&= -2\int_0^{1} (1 - 2t)dt \\
&= 0
\end{aligned}
$$

(c) Parametrize the curve by the concatenation $\mathbf{r}_1(t) = (1 - t, -t),\ t \in [0, 1]$ and $\mathbf{r}_2(t) = (-t, t - 1),\ t \in [0, 1]$. Then $\mathbf{r}_1'(t) = (-1, -1)$ and $\mathbf{r}_2'(t) = (-1, 1)$.

$$
\begin{aligned}
\int_C F \cdot ds &= \int_0^{1} (1 - 2t, -(1 - t)^2 - t^2) \cdot (-1, -1)dt \\
&\quad + \int_0^{1} (-1, -t^2 - (t - 1)^2) \cdot (-1, 1)dt \\
&= \int_0^{1} [2t - 1 + (1 - t)^2 + t^2]dt + \int_0^{1} [1 - t^2 - (t - 1)^2]dt \\
&= 1
\end{aligned}
$$

43.

$$
F(x, y) = \left(-\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}}\right)
$$

51. We have

$$
\begin{cases}
\mathbf{r}_1(t) = (\cos t, \sin t, t) & t \in [0, \frac{\pi}{2}] \\
\mathbf{r}_2(t) = (0, 1, \frac{\pi}{2}(1 - t)) & t \in [0, 1] \\
\mathbf{r}_3(t) = (t, 1 - t, 0) & t \in [0, 1]
\end{cases}
$$

and

$$\begin{cases} \mathbf{r}_1'(t) &= (-\sin t, \cos t, 1) \\ \mathbf{r}_2'(t) &= (0, 0, -\frac{\pi}{2}) \\ \mathbf{r}_3'(t) &= (1, -1, 0). \end{cases}$$

$$\begin{aligned} \text{integral} &= 2\int_0^{\frac{\pi}{2}} (\cos t, t, \sin t) \cdot (-\sin t, \cos t, 1) dt \\ &\quad + 2\int_0^1 \left(0, \frac{\pi}{2}(1-t), 1\right) \cdot \left(0, 0, -\frac{\pi}{2}\right) dt \\ &\quad + 2\int_0^1 (t, 0, 1-t) \cdot (1, -1, 0) dt \\ &= 2\int_0^{\frac{\pi}{2}} (-\sin t \cos t + t \cos t + \sin t) dt + 2\int_0^1 \left(-\frac{\pi}{2}\right) dt + 2\int_0^1 t dt \\ &= 0 \end{aligned}$$