MATH2020A Homework 5

(15.6)

2.

$$I_x = \int_0^3 \int_0^3 \delta y^2 dydx = \int_0^3 \delta \frac{27}{3} dx = 27\delta$$
$$I_y = \int_0^3 \int_0^3 \delta x^2 dydx = \int_0^3 3\delta y^2 dx = 27\delta$$

6. Assume density is 1. So we have mass

$$M = \int_0^\pi \int_0^{\sin x} dydx = \int_0^\pi \sin x dy = 2$$

and first moments

$$M_x = \int_0^\pi \int_0^{\sin x} ydydx = \int_0^\pi \frac{\sin^2 x}{2} dx = \frac{\pi}{4}$$
$$M_y = \int_0^\pi \int_0^{\sin x} xdydx = \int_0^\pi x \sin x dx = -\pi + \int_0^\pi \cos x dx = \pi$$

Hence the centroid is $(\frac{M_y}{M}, \frac{M_x}{M}) = (\frac{\pi}{2}, \frac{\pi}{8})$

9. Assume density is 1. The mass is

$$M = \int_{-\infty}^0 \int_0^{e^x} dydx = \int_{-\infty}^0 e^x dx = 1$$

and first moments are

$$
\begin{aligned}
M_x &= \int_{-\infty}^{0}\int_{0}^{e^x} y\,dy\,dx = \int_{-\infty}^{0} \frac{e^{2x}}{2}dx = \frac{1}{4} \\
M_y &= \int_{-\infty}^{0}\int_{0}^{e^x} x\,dy\,dx = \int_{-\infty}^{0} e^x x\,dx = 0 - \int_{-\infty}^{0} = -1
\end{aligned}
$$

So the centroid is $(-1, \frac{1}{4})$.

13.
The mass is

$$
\begin{aligned}
M &= \int_0^1\int_x^{2-x}(6x+3y+3)dydx \\
&= \int_0^1 6x(2-2x) + \frac{3}{2}[(2-x)^2 - x^2] + 3(2-2x)dx \\
&= \int_0^1 12 - 12x^2 dx \\
&= 8
\end{aligned}
$$

The first moments are

$$
\begin{aligned}
M_x &= \int_0^1\int_x^{2-x}(6x+3y+3)ydydx \\
&= \int_0^1 3x(4-4x) + (8 - 12x + 6x^2 - 2x^3) + 3(2-2x)dx \\
&= \int_0^1 14 - 6x - 6x^2 - 2x^3 dx \\
&= \frac{17}{2} \\
M_y &= \int_0^1\int_x^{2-x}(6x+3y+3)xdydx \\
&= \int_0^1 6x^2(2-2x) + 3x(2-2x) + 3x(2-2x)dx \\
&= \int_0^1 12x - 12x^3 dx \\
&= 3
\end{aligned}
$$

So the mass of center is $(\frac{M_y}{M}, \frac{M_x}{M}) = (\frac{3}{8}, \frac{17}{16})$.

23. Suppose the constant density is δ. The mass is

$$M = \int_{-1}^{1}\int_{-1}^{1}\int_{4y^2}^{4} \delta dz dy dx = \int_{-1}^{1}\int_{-1}^{1} \delta(4-4y^2) dy dx = \int_{-1}^{1} \frac{16\delta}{3} dx = \frac{32\delta}{3}$$

First moments are

$$\begin{aligned}
M_{yz} &= \int_{-1}^{1}\int_{-1}^{1}\int_{4y^2}^{4} x\delta dz dy dx = \int_{-1}^{1} x dx \int_{-1}^{1}\int_{4y^2}^{4} x\delta dz dy = 0 \\
M_{xz} &= \int_{-1}^{1}\int_{-1}^{1}\int_{4y^2}^{4} y\delta dz dy dx = \int_{-1}^{1}\int_{-1}^{1} \delta y(4-4y^2) dy dx = \int_{-1}^{1} 0 dx = 0 \\
M_{xy} &= \int_{-1}^{1}\int_{-1}^{1}\int_{4y^2}^{4} z\delta dz dy dx \int_{-1}^{1}\int_{-1}^{1} \delta(8-8y^4) dy = \int_{-1}^{1} \frac{64\delta}{5} = \frac{128\delta}{5}
\end{aligned}$$

So the center of mass is $(0, 0, \frac{12}{5})$.

For the second moments (Moments of inertia)

$$\begin{aligned}
I_x &= \int_{-1}^{1}\int_{-1}^{1}\int_{4y^2}^{4} \delta(y^2+z^2) dz dy dx = \delta \int_{-1}^{1}\int_{-1}^{1} 4y^2 - 4y^4 + \frac{64-64y^6}{3} dy dx \\
&= \delta \int_{-1}^{1} \frac{8}{3} - \frac{8}{5} + \frac{128}{3} - \frac{128}{21} dx = \frac{7904}{105}\delta \\
I_y &= \int_{-1}^{1}\int_{-1}^{1}\int_{4y^2}^{4} \delta(x^2+z^2) dz dy dx = \delta \int_{-1}^{1}\int_{-1}^{1} x^2(4-4y^2) + \frac{64-64y^6}{3} dy dx \\
&= \delta \int_{-1}^{1} x^2(8-\frac{8}{3}) + \frac{128}{3} - \frac{128}{21} dx = \frac{4832}{63}\delta \\
I_z &= \int_{-1}^{1}\int_{-1}^{1}\int_{4y^2}^{4} \delta(x^2+y^2) dz dy dx = \delta \int_{-1}^{1}\int_{-1}^{1} x^2(4-4y^2) + 4y^2 - 4y^4 dy dx \\
&= \delta \int_{-1}^{1} x^2(8-\frac{8}{3}) + \frac{8}{3} - \frac{8}{5} dx = \frac{256}{45}\delta
\end{aligned}$$

(15.8)

3. a) Compute

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}$$

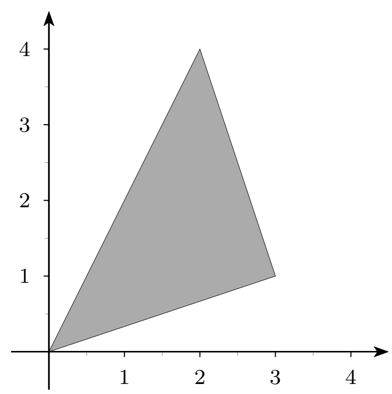
So we have $x = \frac{2u-v}{5}, y = \frac{-u+3v}{10}$. And Jacobi is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{2}{5} \times \frac{3}{10} - \frac{1}{10} \times \frac{1}{5} = \frac{1}{10}$$

b) Original region is $\{(x,y)|x \geq 0, y \geq 0, x+y \leq 0\}$. Substitute u, v and we get

$$\begin{aligned} x \geq 0 &\implies 2u - v \geq 0 \implies 2u \geq v \\ y \geq 0 &\implies -u + 3v \geq 0 \implies 3v \geq u \\ x + y \leq 1 &\implies \frac{3u+v}{10} \leq 1 \implies 3u + v \leq 10 \end{aligned}$$

So our new region is $\{(u,v)|2u \geq v, 3v \geq u, 3u + v \leq 10\}$.



7. This region is given by $R = \{(x,y)|y \leq -\frac{3}{2}x + 3, y \geq -\frac{3}{2}x + 1, y \leq -\frac{x}{4} + 1, y \geq 0\}$. (We do not use the line $y = -\frac{1}{4}x$ here)

Or you can write it as

$$R = \{(x,y)|2 \leq 2y + 3x \leq 6, 4y + x \leq 4, y \geq 0\}$$

Put u, v inside, we will get

$$R = \{(u,v)|2 \leq u \leq 6, v \leq 4, 3v \geq u\}$$

Notice $(3x^2+14xy+8y^2) = (3x+2y)(x+4y) = uv$. After changing variables, we get

$$\begin{aligned}\iint_R (3x^2+14xy+8y^2)dxdy &= \int_2^6 \int_{\frac{u}{3}}^4 uv\frac{1}{10}dvdu \\ &= \int_2^6 u(8-\frac{u^2}{18})\frac{1}{10}du \\ &= \frac{496}{45}\end{aligned}$$

11. We use

$$\begin{cases} x &= ar\cos\theta \\ y &= br\sin\theta \end{cases}$$

So we have

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\begin{array}{cc} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{array}\right| = abr$$

Hence

$$\begin{aligned} I_0 &= \iint_R (x^2+y^2)\delta dxdy = \delta\int_0^{2\pi}\int_0^1 r^2(a^2\cos^2\theta + b^2\sin^2\theta)abrdrd\theta \\ &= \delta\int_0^{2\pi}\frac{1}{4}(a^2\cos^2\theta+b^2\sin^2\theta)d\theta = \frac{(a^2+b^2)ab\pi\delta}{4}\end{aligned}$$

19. Based on our definition, we have

$$\begin{cases} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{cases}$$

Hence

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \left|\begin{array}{ccc} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{array}\right| \\ &= \rho^2\sin\phi\cos^2\phi\cos^2\theta + \rho^2\sin\phi\cos^2\phi\sin^2\theta \\ &\quad +\rho\sin^3\phi\sin^2\theta + \rho^2\sin^3\phi\cos^2\phi \\ &= \rho^2\sin\phi\end{aligned}$$

25. By symmetric properties of semielliposoid with x, y axis, we know its center of mass lies on the z-axis.

Based on the hint(i.e. the centroid of a solid hemisphere lies on the axis of symmetry three eighths of the way from the base toward the top), we have following result,

$$\frac{3}{8} = \iiint\limits_{\{(x,y,z)|x^2+y^2+z^2\leq 1, z\geq 0\}} z dz dy dx \Bigg/ \iiint\limits_{\{(x,y,z)|x^2+y^2+z^2\leq 1, z\geq 0\}} dz dy dx$$

So for semielliposoid, we choose $x = au, y = bv, z = cw$, and find

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = abc$$

Hence

$$\begin{aligned}
& \iiint\limits_{\{(x,y,z)|(\frac{x}{a})^2+(\frac{y}{b})^2+(\frac{z}{c})^2\leq 1, z\geq 0\}} z dz dy dz \Bigg/ \iiint\limits_{\{(x,y,z)|(\frac{x}{a})^2+(\frac{y}{b})^2+(\frac{z}{c})^2\leq 1, z\geq 0\}} dz dy dz \\
= & \iiint\limits_{\{(u,v,w)|u^2+v^2+w^2\leq 1, w\geq 0\}} cwabc dw dv du \Bigg/ \iiint\limits_{\{(u,v,w)|u^2+v^2+w^2\leq 1, w\geq 0\}} abc dw dv du \\
= & \ c \iiint\limits_{\{(u,v,w)|u^2+v^2+w^2\leq 1, w\geq 0\}} w dw dv du \Bigg/ \iiint\limits_{\{(u,v,w)|u^2+v^2+w^2\leq 1, w\geq 0\}} dw dv du \\
= & \ \frac{3c}{8} \quad \text{(based on first equality)}
\end{aligned}$$

Hence the center of mass is $(0, 0, \frac{3}{8}c)$.