Math 2230B, Complex Variables with Applications

For the functions f and contours C in Exercise 1 through 3, use parametric representations for C, or legs of C, to evaluate

$$\int_C f(z)dz.$$

1. f(z) is the principal branch

$$z^{-1-2i} = \exp[(-1-2i)\text{Logz}] \quad (|z| > 0, -\pi < Argz < \pi)$$

   of the indicated power function, and C is the contour

$$z = e^{i\theta} \quad (0 \leq \theta \leq \frac{\pi}{2}).$$

2. f(z) is the principal branch

$$z^{a-1} = \exp[(a-1)\text{Logz}] \quad (|z| > 0, -\pi < Argz < \pi)$$

   of the power function z^{a-1}, where a is a nonzero real number, and C is the positively oriented circle of radius R about the origin.

3. Let C denote the positively oriented unit circle $|z| = 1$ about the origin

   (a) Show that if f(z) is the principal branch

$$z^{-3/4} = \exp[-\frac{3}{4}Logz] \quad (|z| > 0, -\pi < Argz < \pi)$$

   of $z^{-3/4}$, then

$$\int_C f(z)dz = 4\sqrt{2}i.$$

   (b) Show that if g(z) is the branch

$$z^{-3/4} = \exp[-\frac{3}{4}logz] \quad (|z| > 0, 0 < Argz < 2\pi)$$

   of the same function as in part (a), then

$$\int_C g(z)dz = -4 + 4i.$$

4. Without evaluating the integral, show that

   (a) $|\int_C \frac{z+4}{z^3-1}dz| \leq \frac{6\pi}{7}$;

(b) $|\int_C \frac{dz}{z^2-1}| \leq \frac{\pi}{3}$

when C is the arc of the circle $|z| = 2$ from z=2 to z=2i that lies in the first quadrant.

5. Let C denote the line segment from z=i to z=1, and show that

$$|\int_C \frac{dz}{z^4}| \leq 4\sqrt{2}$$

without evaluating the integral.

6. Let C_R be the circle $|z| = R(R > 1)$, decribed in the counterclockwise direction. Show that

$$|\int_{C_R} \frac{\text{Log z}}{z^2} dz| < 2\pi(\frac{\pi + \ln R}{R}),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

7. Let C_ρ denote a circle $|z| = \rho(0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that f(z) is analytic in the disk $|z| \leq 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, independent of ρ, such that

$$|\int_{C_\rho} z^{-1/2} f(z) dz| \leq 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Suggestion: Note that since f(z) is analytic, and therefore continues, throughout the disk $|z| \leq 1$, it is bounded there (Sec. 18).