Math 2230A, Complex Variables with Applications

1. Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if
   (a) $f(z) = \overline{z}$; (b) $f(z) = z - \overline{z}$
   (c) $f(z) = 2x + ixy^2$; (d) $f(z) = e^x e^{-iy}$

2. Use the theorem in Sec. 23 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when
   (a) $f(z) = iz + 2$; (b) $f(z) = e^{-x} e^{-iy}$
   (c) $f(z) = z^3$; (d) $f(z) = \cos x \cosh y - i \sin x \sinh y$

3. From results obtained in Secs. 21 and 23, determine where $f'(z)$ exists and find its value when
   (a) $f(z) = 1/z$; (b) $f(z) = x^2 + iy^2$; (c) $f(z) = z \operatorname{Im} z$

4. Apply the theorem in Sec. 23 to verify that each of these functions is entire:
   (a) $f(z) = 3x + y + i(3y - x)$; (b) $f(z) = \cosh x \cos y + i \sinh x \sin y$
   (c) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$; (d) $f(z) = (z^2 - 2)\, e^{-x} e^{-iy}$

5. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

   (a) $f(z) = xy + iy$; (b) $f(z) = 2xy + i\left(x^2 - y^2\right)$; (c) $f(z) = e^y e^{ix}$

6. Let a function f be analytic everywhere in a domain D. Prove that if f(z) is real-valued for all z in D, then f(z) must be constant throughout D.

7. Evaluate the following integrals:

   (a) $\int_0^1 (1 + it)^2 dt$;
   (b) $\int_1^2 (\frac{1}{t} - i)^2 dt$;
   (c) $\int_0^{\frac{\pi}{6}} e^{i2t} dt$;
   (d) $\int_0^{\infty} e^{-zt} dt \quad (Rez > 0)$.

8. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when m} \neq \text{n.} \\ 2\pi & \text{when m=n.} \end{cases}$$

9. According to definition (2), Sec. 42, of definite integrals of complex-valued functions of a real variavble,

$$\int_0^\pi e^{(1+i)x}dx = \int_0^\pi e^x \cos x dx + i \int_0^\pi e^x \sin x dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

For the function f and contour C in exercise 10 through 15, us parametric representation for C, or legs of C, to evaluate

$$\int_C f(z)dz.$$

10. $f(z) = (z+2)/z$ and C is
    (a) the semicircle $z = 2e^{i\theta}(0 \le \theta \le \pi)$;
    (b) the semicircle $z = 2e^{i\theta}(\pi \le \theta \le 2\pi)$;
    (c) the circle $z = 2e^{i\theta}(0 \le \theta \le 2\pi)$.
11. $f(z) = z - 1$ and C is the arc from z=0 to z=2 consisting of
    (a) the semicircle $z = 1 + e^{i\theta}(\pi \le \theta \le 2\pi)$;
    (b) the segment $z = x(0 \le x \le 2)$ of the real axis.
12. $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points 0, 1, 1+i, and i, the orientation of C being in the counterclockwise direction.
13. $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

14. $f(z) = 1$ and C is an arbitrary contour from any fixed point z_1 to any fixed point z_2 in the z plane.
15. f(z) is the principal branch

$$z^i = \exp(i\text{Logz}) \quad (|z| > 0, -\pi < Argz < \pi)$$

of the power function z^i, and C is the semicircle $z = e^{i\theta}(0 \le \theta \le \pi)$.