Math 2230B, Complex Variables with Applications

1. In each case, find the order m of the pole and the corresponding residue B at the singularity z=0.

$$(a) f(z) = \frac{\sinh z}{z^4}; \quad (b) f(z) = \frac{1}{z\,(e^z - 1)}.$$

2. Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)},$$

taken counter-clockwise around the circle $(a)|z| = 2; (b)|z+2| = 3$.

3. Find the value of the integral

$$\int_C \frac{\cosh \pi z}{z\,(z^2+1)} dz$$

when C is the circle $|z| = 2$, described in the positive sense.

4. Show that

$$\text{(a) } \operatorname{Res}_{z=\pi i/2} \frac{\sinh z}{z^2 \cosh z} = -\frac{4}{\pi^2};$$

$$\text{(b) } \operatorname{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -2\cos(\pi t).$$

5. Show that

$$\text{(a) } \operatorname{Res}_{z=z_n}(z \sec z) = (-1)^{n+1} z_n \quad \text{where} \quad z_n = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots);$$

$$\text{(b) } \operatorname{Res}_{z=z_n}(\tanh z) = 1 \quad \text{where} \quad z_n = \left(\frac{\pi}{2} + n\pi\right) i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

6. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

(a) $\int_C \tan z dz$; (b) $\int_C \frac{dz}{\sinh 2z}$.

7. Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi$$

where N is a positive integer. Show that

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right].$$

Then using the fact that the value of this integral tends to zero as N tends to infinity(Exercise 8, Sec. 47), point out how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

8. Show that

$$\int_C \frac{dz}{(z^2-1)^2 + 3} = \frac{\pi}{2\sqrt{2}},$$

where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2, y = 0$, and $y = 1$.
Suggestion: By observing that the four zeros of the polynomial $q(z) = (z^2-1)^2 + 3$ are the square roots of the number $1 \pm \sqrt{3}i$, show that the reciprocal $\frac{1}{q(z)}$ is analytic inside and on C except at the points

$$z_0 = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\overline{z_0} = \frac{-\sqrt{3}+i}{\sqrt{2}}.$$

Then apply Theorem 2 in Sec.83.