Math 2230B, Complex Variables with Applications

1. Find the residue at z=0 of the function

$$(a)\frac{1}{z+z^2}; \quad (b)z\cos\left(\frac{1}{z}\right); \quad (c)\frac{z-\sin z}{z}; \quad (d)\frac{\cot z}{z^4}; \quad (e)\frac{\sinh z}{z^4(1-z^2)}.$$

2. Use Cauchy's residue theorem(Sec.76) to evaluate the integral of each of these functions around the circle $|z|=3$ in the positive sense:

$$(a)\frac{\exp(-z)}{z^2}; \quad (b)\frac{\exp(-z)}{(z-1)^2}; \quad (c)z^2\exp\left(\frac{1}{z}\right); \quad (d)\frac{z+1}{z^2-2z}.$$

3. In the example in Sec.76, two residues were used to evaluate the integral

$$\int_C \frac{4z-5}{z(z-1)}dz$$

where C is the positively oriented circle $|z|=2$. Evaluate this integral once again by using the theorem in Sec.77 and finding only one residue.

4. Use the theorem in Sec.77, involving a single residue, to evaluate the integral of each of these functions around the circle $|z|=2$ in the positive sense:
(a) $\frac{z^5}{1-z^3}$; (b) $\frac{1}{1+z^2}$; (c) $\frac{1}{z}$.

5. Let C denote the circle $|z|=1$, taken counterclockwise, and use the following steps to show that

$$\int_C \exp\left(z+\frac{1}{z}\right)dz = 2\pi i\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}.$$

   (a) By using the Maclaurin series for e^z and referring to Theorem 1 in Sec.71, which justifies the term by term integration that is to be used, write the above integral as

$$\sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n\exp\left(\frac{1}{z}\right)dz.$$

   (b) Apply the theorem in Sec.76 to evaluate the integrals appearing in part (a) to arrive at the desired result.

6. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

$$(a)z\exp\left(\frac{1}{z}\right); \quad (b)\frac{z^2}{1+z}; \quad (c)\frac{\sin z}{z}; \quad (d)\frac{\cos z}{z}; \quad (e)\frac{1}{(2-z)^3}.$$

7. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

$$(a)\frac{1-\cosh z}{z^3}; \quad (b)\frac{1-\exp(2z)}{z^4}; \quad (c)\frac{\exp(2z)}{(z-1)^2}.$$

8. Suppose that a function f is analytic at z_0, and write $g(z) = \frac{f(z)}{(z-z_0)}$. Show that

   (a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

   (b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

   Suggestion: As pointed out in Sec. 62, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.