Math 2230,Complex Variables with Applications

1. In each case, write the function $f(z)$ in the form $f(z) = u(x,y) + iv(x,y)$:
   (a) $f(z) = z^3 + z + 1$;
   (b) $f(z) = \frac{\overline{z}^2}{z} \quad (z \neq 0)$.

2. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions(see Sec.6)

$$x = \frac{z + \overline{z}}{z} \quad \text{and} \quad \frac{z - \overline{z}}{2i}$$

   to write $f(z)$ in terms of z, and simplify the result.

3. Write the function

$$f(z) = z + \frac{1}{z} \quad (z \neq 0)$$

   in the form $f(z) = u(r,\theta) + iv(r,\theta)$.

4. Use definition (2), Sec. 15, of limit to prove that
   (a) $\lim_{z\to z_0} Rez = Rez_0$;
   (b) $\lim_{z\to z_0} \overline{z} = \overline{z_0}$;
   (c) $\lim_{z\to 0} \frac{\overline{z}^2}{z} = 0$.

5. Show that the function

$$f(z) = (\frac{z}{\overline{z}})^2$$

   has the value 1 at all nonzero points on the real and imaginary axes, where $z = (x,0)$ and $z = (0,y)$, respectively, but that it has the value -1 at all nonzero points on the line y=x, where $z = (x,x)$. Thus show that the limit of $f(z)$ as z tends to 0 does not exist.

6. Use definition (2), Sec. 15, of limit to prove that

$$\text{if} \lim_{z\to z_0} f(z) = \omega_0, \text{then} \lim_{z\to z_0} |f(z)| = |\omega_0|.$$

7. With the aid of the theorem in Sec. 17, show that when

$$T(z) = \frac{az + b}{cz + d} \quad (ad - bc \neq 0),$$

   (a) $\lim_{z\to\infty} T(z) = \infty$ if c=0;

(b) $\lim_{z\to\infty} T(z) = \frac{a}{c}$ and $\lim_{z\to -d/c} T(z) = \infty$ if $c \neq 0$.

8. Use definition (3), Sec. 19, to give a direct proof that

$$\frac{d\omega}{dz} = 2z \quad \text{when} \quad \omega = z^2.$$

9. Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point z when

   (a) $f(z) = Rez$;

   (b) $f(z) = Imz$.

10. Let f denote the function whose values are

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{when} \quad z \neq 0 \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Show that if z=0, then $\Delta\omega/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the $\Delta z$, or $\Delta x \Delta y$, plane. Then show that $\Delta\omega/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in that plane(Fig.29). Conclude from these observations that $f'(0)$ does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the $\Delta z$ plane.

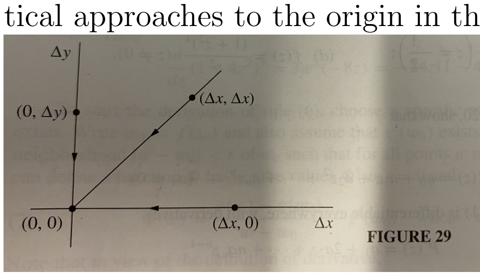

FIGURE 29

11. Show that

   (a) $(1+i)^i = \exp(-\frac{\pi}{4} + 2n\pi)\exp(i\frac{ln2}{2}) \quad (n = 0, \pm1, \pm2, ...)$;

   (b) $\frac{1}{i^{2i}} = \exp[(4n+1)\pi] \quad (n = 0, \pm1, \pm2, ...)$.

12. Find the principal value of

   (a) $(-i)^i$;

   (b) $[\frac{e}{2}(-1-\sqrt{3}i)]^{3\pi i}$;

   (c) $(1-i)^{4i}$.

13. Use definition (1), Sec 35, of $z^c$ to show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

14. Show that if $z \neq 0$ and a is a real number, then $|z^a| = \exp(aln|z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

15. With the aid of expression (14), Sec. 37, show that the roots of the equation $\cos z = 2$ are

$$z = 2n\pi + i\cosh^{-1} 2 \quad (n = 0, \pm 1, \pm 2, ...).$$

Then express them in the form

$$z = 2n\pi \pm i ln(2 + \sqrt{3}) \quad (n = 0, \pm 1, \pm 2, ...).$$