THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Tutorial 5

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Theorem 1. *(Cauchy-Goursat theorem) If $f(z)$ is analytic at all points interior to and on a simple closed contour C (the closure of the bounded component divided by the contour), then*

$$\int_C f(z)dz = 0$$

Remark: You may use Cauchy Riemann equation to check the analyticity. Or you may just see if the function is composed by some elementary analytic functions. (polynomial, trigonometric function, exponential function...)

Remark: Generally, analyticity is not equivalent to having an antiderivative, so this theorem is sightly different with theorem 1 in tutorial 4.

Theorem 2. *Suppose that*

1. *C is simply closed contour in counterclockwise direction;*
2. *C_k(n=1,..,n) are simply closed contour interior to C, all in clockwise direction, that are disjoint and whose interiors have no common points.*

If f is analytic on all of the contour C and C_k and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k, then

$$\int_C f dz + \sum_1^n \int_{C_k} f dz = 0$$

Remark : You should draw a diagram about it.

Remark : $\int_C f dz = -\int_{-C} f dz$ where C is in counterclockwise direction and $-C$ is in clockwise direction.

Remark : You can replace the contour C with a circle or other "simple" contour in most of the case.

Theorem 3. *(Cauchy Integral Formula) Let f be analytic inside and on a simple closed contour C. If z_0 is interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0}$$

Remark : You can see that an analytic function is uniquely determined by its boundary value. (compare with the case of real variable function)

Lemma 1. *Let h be continuous on a simple closed contour C. Define $H_n(z) = \int_C \frac{h(w)dw}{(w-z)^n}$ for $n \geq 1$ and z being inside the interior of C. Then H_n is analytic inside the interior of C and $H_n'(z) = nH_{n+1}(z)$.*

Using this lemma, we have:

Theorem 4. *(Generalized Cauchy Integral Formula) Let f be analytic inside and on a simple closed contour C. If z_0 is interior to C, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}$$

Remark : This is why analyticity implies complex infinite differentiability.

Exercise:

1. Use theorem 1 to show that the integrals are zero along the contour $|z| = 1$,
   (a) $\displaystyle\int_C \frac{dz}{z^2 + 2z + 2}$ (b) $\displaystyle\int_C Log(z + 2)dz$.

2. Find $\displaystyle\int_C \frac{dz}{z^2 + 4}$ where C represents the circle $|z - i| = 2$.

3. Find $\displaystyle\int_C \frac{\cos z dz}{z(z^2 + 2)}$ where C represents the square whose sides lie along $x = \pm 2$ and $y = \pm 2$.