Hw 9

① we diff. the power series term by term since the series converges absolutely and uniformly on any compact subset of $\{|z|<1\}$ due to ratio test. The second result is obtained by the same way.

② obvious.

③ We assume that $\left|\frac{z-2}{2}\right| < 1$, then

$$\frac{1}{z} = \frac{1}{2}\left(\frac{1}{1+(z-2)/2}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z-2}{2}\right)^n$$ by sum of G.S.

The second result is obtained by the same way in ①.

④ For $z \neq 0$, we express,

$$\frac{1-\cos z}{z^2} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(n+2)!}$$ by the taylor series of $\cos z$

The series converges for all $z \neq 0$ since $\frac{1-\cos z}{z^2}$ is analytic for all $z \neq 0$. Also the series converges clearly for $z=0$ and takes the value $1/2$. Thus the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(n+2)!}$$

equals to $f(z)$ and converges $\forall z$ and hence entire.

(5) Since $f(z)$ is analytic far away from $z = \pm \pi/2$, we assume that $|z - \pi/2| < 1/2$ first.

Also, $\dfrac{\cos z}{z^2 - (\pi/2)^2} = \dfrac{1}{z + \pi/2}\left(\dfrac{\cos z}{z - \pi/2}\right)$ and

$\dfrac{1}{z + \pi/2}$ is analytic for $|z - \pi/2| < 1/2$, it is enough to study $\dfrac{\cos z}{z - \pi/2}$. Since

$$\cos z = -\sin(z - \pi/2) = \sum_{n=0}^{\infty} (-1)^n \frac{(z - \pi/2)^{2n+1}}{(2n+1)!}$$

Then for $z \neq \pi/2$,

$$\frac{\cos z}{z - \pi/2} = \sum_{n=0}^{\infty} (-1)^n \frac{(z - \pi/2)^{2n}}{(2n+1)!}.$$

Since $\dfrac{\cos z}{z - \pi/2}$ is analytic in $|z - \pi/2| < 1/2$ by the result in (4), thus $\dfrac{\cos z}{z^2 - (\pi/2)^2}$ is also analytic in $|z - \pi/2| < 1/2$.

Similarly, $\dfrac{\cos z}{z^2-(\pi/2)^2}$ is also analytic in $|z+\pi/2|<\frac{1}{2}$.

Then it is done.

(6) Obvious.

(7) Similar to (4), (5) by using (6).

(8) Similar to (4), (5) by expressing

$$f(z) = \sum_{n=m+1} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \quad \text{for } z \neq z_0 .$$

(9) let $g(z)S_2(z) = \sum_{n\geq 0}^{N-1} b_n g(z)\dfrac{1}{(z-z_0)^n} + g(z)\rho_N(z)$

$\rho_N(z) = \sum_{n\geq N}^{\infty} \dfrac{b_n}{(z-z_0)^n}$ which converges absolutely and uniformly on C. Let M be the max. value of $|g(z)|$ on C and L denote the length of C. Due to the uniform conv. of $\sum_{n\geq 1}^{\infty} \dfrac{b_n}{(z-z_0)^n}$ on C, ~~ε~~

~~, There is $N_\varepsilon \in \mathbb{N}$ s.t. $\forall N > N_\varepsilon$~~

$|\rho_N| = \left| \sum_{n\geq N}^{\infty} \dfrac{b_n}{(z-z_0)^n} \right| \to 0$ ~~Always~~ as $N \to \infty$, then

$\int_C g S_2 = \sum_{n\geq 1}^{N-1} b_n \int_C g(z)\dfrac{1}{(z-z_0)^n} + \int g \rho_N$

taking $N \to \infty$, $\int_C g S_2 = \sum_{n\geq 1}^{\infty} b_n \int_C g \dfrac{b_n}{(z-z_0)^n}$.

The final result is obtained by combining both results.