THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Tutorial 2
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0.1 Polynomial and Rational function

The **domain of definition** (or simply the domain) of a function is the set of input for which the function value is defined.

Definition 1. *We call the function in the form of*

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + .. + a_n z^n \text{ for } n = 0, 1, 2, ... \text{ and } a_n \neq 0$$

to be the polynomial of degree n.

Remark: The domain of definition of polynomial is clearly $\mathbb{C}$.
Remark: The condition of $a_n \neq 0$ is meaningful. Otherwise, the degree of the polynomial could be smaller than n.

Definition 2. *Given two polynomials $P_n(z)$ and $Q_m(z)$, the function $R(z) = \dfrac{P_n(z)}{Q_m(z)}$ is called the* ***rational function****.*

Remark: The domain of definition of rational function is clearly $\mathbb{C} \setminus \{z_1, z_2, ..., z_m\}$ where $z_1, z_2, ..., z_m$ are the roots of $Q_m(z) = 0$.

0.2 Trigonometric function

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = -i \sin(iz) = \frac{e^{z} - e^{-z}}{2} \qquad \cosh z = \cos(iz) = \frac{e^{z} + e^{-z}}{2}$$

Remark: The domain of definition of these trigonometric functions are clearly $\mathbb{C}$ since that of exponential function is also $\mathbb{C}$.

0.3 Logarithmic function

If the logarithmic is the "inverse" of exponential function, we let $z_0 = re^{i\theta} \neq 0$ for $-\pi < \theta \leq \pi$ ($\theta = Arg(z)$) and we expect to have

$$\log(z_0) = \log(re^{i\theta}) = \log(r) + i\theta$$

However, $z_0 = re^{i\theta} = re^{i\theta + 2k\pi i}$ for any integers k, hence we have

$$\log(z_0) = \log(re^{i\theta + 2k\pi i}) = \log(r) + i\theta + 2k\pi i$$

It shows that the logarithmic defined in this way is not a function. We see that $\log(z_0)$ represent a set

$$\log(z_0) := \{ \log(r) + i\theta + 2k\pi i \mid k \in \mathbb{Z} \}$$

or we will call log is a multiple-valued function. It is quite similar to the definition of argument of a complex number. Therefore we should define logarithmic in a unique way.

Definition 3. *The principal value of $\log(z)$ equals to*

$$Log(z) = \log|z| + iArg(z)$$

where $-\pi < Arg(z) \leq \pi$.

Remark 1 : Since it is reasonable to define the range of the angle of z_0 in another way, say, $z_0 = re^{i\theta} \neq 0$ for $a < \theta \leq 2\pi + a$ for any real number a. Such a choice of range of the angle of z is called **branch**. And we can define another single-value function for log by $\log(r) + i\theta$ with $a < \theta \leq 2\pi + a$. (The word "principal" in definition 3 means that $a = -\pi$. We would not call the single-valued log to be principal if $a \neq 0$.) And the range $-\pi < \theta \leq \pi$ is called **principal branch**. The straight line $\{z \in \mathbb{C} : Arg(z) = a + 2\pi\}$ is called the **branch cut**.

Remark 2 : The domain of definition of $Log(z)$ is $\mathbb{C} \setminus \{0\}$ because of $\log |z|$.

Remark 3 : Although $Log(z)$ can be defined on the ray $\theta = a$, $Log(z)$ is not continuous there (not analytic).

Remark 4 : $Log(z)$ is analytic in the domain $r > 0$ and $-\pi < Arg(z) < \pi$ (or other branch).

0.4 Power function

Definition 4. *Let $z \neq 0$ and $c \in \mathbb{C}$, the power is defined as*

$$z^c = e^{c\ Log(z)}$$

Clearly it can be defined for other branch.

Remark : The domain of definition of power function is again $\mathbb{C} \setminus \{0\}$.
Remark : Some operations which is true in real number turn out is false in complex number:
(a) $z^{c_1} z^{c_2} = z^{c_1+c_2}$; (b) $(z^{c_1})^{c_2} \neq z^{c_1 c_2}$; (c) $(zw)^c \neq z^c w^c$.

0.5 Continuity of a Function

Definition 5. *Let Ω be open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$. Let $z_0 \in \Omega$, we say $\lim_{z \to z_0} f(z) = c$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - c| < \varepsilon$.*

Proposition 1. *Let Ω be open subset of $\mathbb{C}$ and $f = f_1 + if_2 : \Omega \to \mathbb{C}$. Let $z_0 \in \Omega$, then f is continuous at z_0 if and only if f_1 and f_2 are continuous at z_0. In other words, $\lim_{z \to z_0} f(z) = f(z_0)$ if and only if $\lim_{z \to z_0} f_1(z) = f_1(z_0)$ and $\lim_{z \to z_0} f_2(z) = f_2(z_0)$.*

0.6 Exercise

1. Compute the value of $\log(-1 + \sqrt{3}i)$ with branch $-\pi < Arg(z) \leq \pi$.
2. Find the domain of $f(z) = Log(z - i)$.
3. Find the principal values of $(1 + i)^i$.
4. Describe the image under $f = e^z$ of the following sets:
(a) The set of $z = x + yi$ such that $x \leq 1$ and $0 \leq y \leq \pi$.
(b) The set of $z = x + yi$ such that $0 \leq y \leq \pi$.