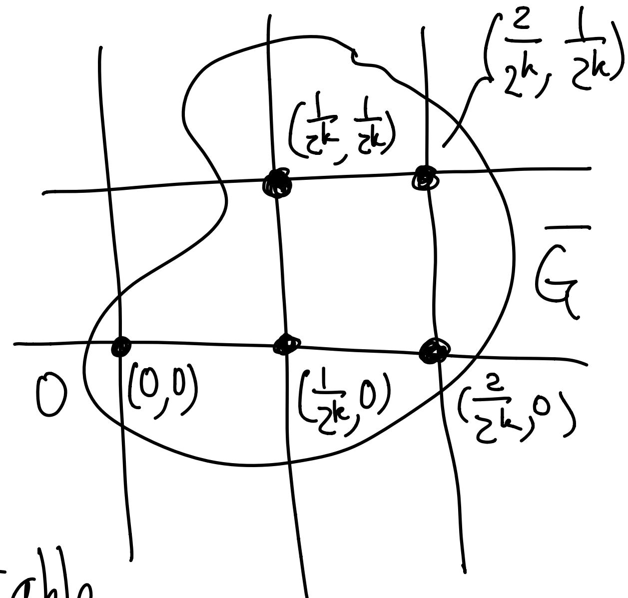
Thm 4.2 (Ascoli's Theorem)

Suppose that G is a bounded nonempty open set in $\mathbb{R}^m$. Then a set $\mathcal{E} \subset C(\overline{G}) \ (= C_b(\overline{G}))$ is precompact iff $\mathcal{E}$ is bounded (in supnorm) and equicontinuous.

Pf: Define $E = \bigcup_{k=0}^{\infty} E_k$, where

$$E_k = \left\{ x = \frac{1}{2^k}\begin{pmatrix} l_1 \\ \vdots \\ l_m \end{pmatrix} \in \overline{G} : l_i \in \mathbb{Z},\ i = 1, \cdots, m \right\}.$$



Then $\overline{G}$ closed and bounded $\Rightarrow$

E_k is finite.

Hence $E = \bigcup_k E_k$ is countable.

Let $\{f_n\}$ be a sequence in $\mathcal{E}$. Then $\mathcal{E}$ bounded

$\Rightarrow \exists M > 0$ such that $\|f_n\|_\infty \le M, \ \forall n$

i.e. $|f_n(x)| \leq M$, $\forall n$ & $\forall x \in \overline{G}$

In particular, $\forall x \in E$,

$$|f_n(x)| \leq M, \ \forall n.$$

i.e. If we arrange the points of E in a sequence $E = \{z_j\}_{j=1}^{\infty}$, then $\forall j \geq 1$,

$$\{f_n(z_j)\} \text{ is a bounded sequence.}$$

Hence one can apply Lemma 4.3 to find a subsequence $\{g_n\}$ of $\{f_n\}$ (using the same notation "n" for the index) such that $\forall x \in E$, $g_n(x)$ is convergent.

We claim that g_n is the required convergent subsequence of f_n in the metric $(C(\overline{G}), d_\infty)$.

(Note that we only have pointwise convergence for countable many points at this moment.)

Since $(C(\bar{G}), d_\infty)$ is complete, we only need to show that $\{g_n\}$ is a Cauchy sequence in $(C(\bar{G}), d_\infty)$.

By equicontinuity of $\mathcal{E}$, ($\Rightarrow$ equicontinuity of $\{g_n\}$)
$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

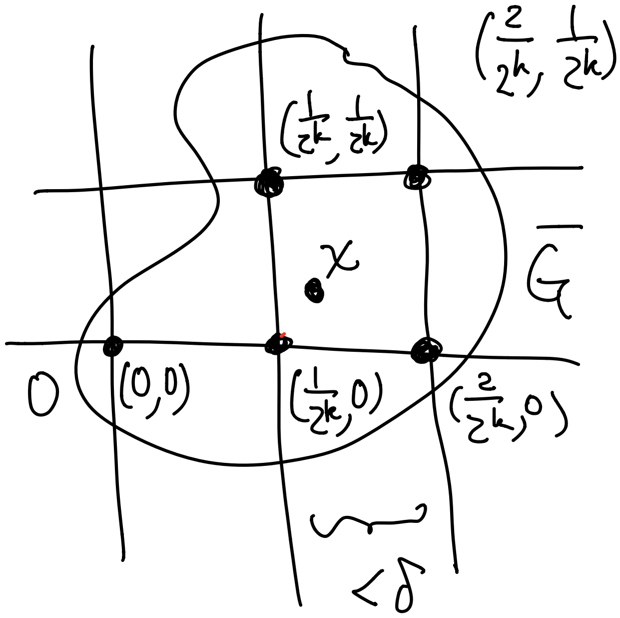
$$|g_n(x) - g_n(y)| < \frac{\varepsilon}{3}, \quad \forall n \ \& \ \forall x, y \in \bar{G} \text{ with } |x-y| < \delta.$$

Note that if k satisfies $\frac{1}{2^k} < \delta$,

then $\forall x \in \bar{G}$, $\exists z_j \in E_k$

such that $|x - z_j| < \delta$. (See figure)

and hence $\quad |g_n(x) - g_n(z_j)| < \frac{\varepsilon}{3}$.



Therefore,

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(z_j)| + |g_n(z_j) - g_m(z_j)|$$
$$+ |g_m(z_j) - g_m(x)|$$

$$< \frac{2\varepsilon}{3} + |g_n(z_j) - g_m(z_j)|.$$

Since $\{g_n(z_j)\}$ is convergent, $\exists\, n_0 = n_0(z_j) \geq 0$

$$|g_n(z_j) - g_m(z_j)| < \frac{\varepsilon}{3}, \quad \forall\, n, m \geq n_0(z_j).$$

$$\Rightarrow \quad |g_n(x) - g_m(x)| < \varepsilon, \quad \forall\, n, m \geq n_0(z_j).$$

(z_j depends on x)

Now take $N_0 = \max_{z_j \in E_k} n_0(z_j) \geq 0$, $\leftarrow$ finite set

then $\forall\, x \in \overline{G}$, we have

$$|g_n(x) - g_m(x)| < \varepsilon, \quad \forall\, n, m \geq N_0.$$

i.e. $$\|g_n - g_m\|_\infty < \varepsilon, \quad \forall\, n, m \geq N_0.$$

This completes the proof of the Theorem.

\#

<u>Remarks</u>

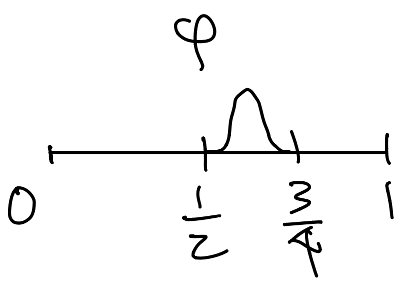
(1) Ascoli's Theorem remains valid for bounded and equicontinuous subsets of $C(G)$.
(i.e. No need to take closure.)
It is because "equicontinuous" $\Rightarrow$ "uniform continuous on G", and then can be extended to uniform continuous on $\overline{G}$. (Details omitted.)

(2) However, <u>boundedness</u> of the domain $\overline{G}$ cannot be removed:

<u>Eg(?)</u> Let $\overline{G} = [0, \infty) \subset \mathbb{R}$.

Take a $\varphi \in C^1[0,1]$ such that

$\varphi \not\equiv 0$ and $\varphi(x) = 0$ on $[0,1] \setminus [\frac{1}{2}, \frac{3}{4}]$



and define

$$f_n(x) = \begin{cases} \varphi(x-n), & \text{if } x \in [n, n+1] \\ 0, & \text{otherwise.} \end{cases}$$

Then one can easily check that

$$f_n \in C(\overline{G}) \quad (\text{in fact } f_n \in C^1(\overline{G}))$$

and $\|f_n\|_{\infty, \overline{G}} = \|\varphi\|_{\infty, [0,1]} > 0$ (and a fixed constant)

$\therefore \ \mathcal{E} = \{f_n\}$ is bounded subset in $C(\overline{G})$.

By Chain rule, $\|\frac{df_n}{dx}\|_{\infty,\bar{G}} = \|\frac{d\varphi}{dx}\|_{\infty,[0,1]} > 0$.

Hence Prop 4.1 implies that

$\mathcal{E} = \{f_n\}$ is also equicontinuous.

Suppose $\exists$ subsequence $\{f_{n_j}\}$ of $\{f_n\}$ converges to some $f \in C(\bar{G})$ in d_∞.

i.e. $f_{n_j} \to f$ uniformly on $\bar{G}$

$\Rightarrow$ pointwise convergence $f_{n_j}(x) \to f(x)$, $\forall x \in \bar{G}$.

However, for fixed x, $f_n(x) = 0$, $\forall n \geq x$, we must have

$$\lim_{j \to +\infty} f_{n_j}(x) = 0.$$

$\therefore$ $f(x) = 0$, $\forall x \in \bar{G}$.

This is a contradiction, since

$0 < \|\varphi\|_{\infty,[0,1]} = \|f_{n_j}\|_{\infty,\bar{G}} = \|f_{n_j} - f\|_{\infty,\bar{G}} \to 0$.

Hence $\mathcal{E}$ is bounded and equicontinuous, but Ascoli's Theorem doesn't hold. #

Converse to Ascoli's Theorem:

Thm 4.4 (Arzela's Theorem)

Suppose that G is a bounded nonempty open set in $\mathbb{R}^m$. Then every precompact set in $C(\overline{G})$ must be bounded and equicontinuous.

Pf: Let $\mathcal{E} \subset C(\overline{G})$ be precompact.

If $\mathcal{E}$ is unbounded, then $\exists f_n \in \mathcal{E} \subset C(\overline{G})$ such that $\lim_{n \to +\infty} \|f_n\|_\infty = \infty$.

Then this subset $\{f_n\}$ of $\mathcal{E}$ cannot contain any convergent subsequence. This contradicts the precompactness. Hence $\mathcal{E}$ must be bounded.

Now suppose on the contrary that $\mathcal{E}$ is precompact, bounded but not equicontinuous.

Then $\exists\ \varepsilon_0 > 0$ such that $\forall\ \delta > 0$ $\exists\ x, y \in \overline{G}$ and $f \in \mathcal{E}$ satisfying

$$|f(x)-f(y)| \geq \varepsilon_0 \quad \& \quad d(x,y) < \delta.$$

In particular, by choosing $\delta = \frac{1}{n} > 0$, for $n = 1, 2, \ldots$

$\exists\ x_n, y_n \in \overline{G}$ and $f_n \in \mathcal{E}$ satisfying

$$|f_n(x_n) - f_n(y_n)| \geq \varepsilon_0 \quad \& \quad d(x_n, y_n) < \frac{1}{n}.$$

By precompactness, $\exists$ convergent subseq. $\{f_{n_k}\}$ of $\{f_n\}$. Suppose $f \in C(\overline{G})$ is the limit,

i.e. $\quad d_\infty(f_{n_k}, f) \to 0 \quad$, as $j \to +\infty$.

(i.e. f_{n_j} converges uniformly to f on $\overline{G}$)

Since $\overline{G}$ is closed and bounded, the corresponding sequences of points $\{x_{n_k}\}$ $(\{y_{n_k}\})$ contains convergent subsequence. Denotes the subseq. by $\{x_k\}$ and assume $\quad x_k \to z \in \overline{G}$.

And also denote the corresponding subseq. of $\{y_{n_k}\}$

by $\{y_k\}$, and the corresponding subseq. of $\{f_{n_k}\}$

by $\{g_k\}$. Then $\begin{cases} g_k \to f \text{ in } (C(\bar{G}), d_\infty) \\ x_k \to z \text{ in } \bar{G} \end{cases}$

Since $d(x_n, y_n) < \frac{1}{n} \Rightarrow d(x_k, y_k) \to 0$ as $k \to \infty$

$\Rightarrow y_k \to z \in \bar{G}$ too.

Hence $\forall \varepsilon > 0$, $\exists k_0 \geq 0$ s.t.

$$\|g_k - f\|_\infty < \varepsilon, \quad \forall k \geq k_0.$$

and $\exists k_1 \geq 0$ s.t.

$$\left.\begin{array}{l} |f(x_k) - f(z)| < \varepsilon \\ |f(y_k) - f(z)| < \varepsilon \end{array}\right. \quad \forall k \geq k_1$$

Hence for $k \geq \max\{k_0, k_1\}$,

$$|g_k(x_k) - g_k(y_k)| \leq |g_k(x_k) - f(x_k)| + |f(x_k) - f(y_k)| + |f(y_k) - g_k(y_k)|$$

$$< 2\varepsilon + |f(x_k) - f(y_k)|$$

$$\leq 2\varepsilon + |f(x_k) - f(z)| + |f(z) - f(y_k)|$$

$< 4\varepsilon$

We've show that $\forall \varepsilon > 0$, $\exists\ n_0 = n_{\max\{k_0, k_1\}} \geq 0$ such that

$$|f_{n_k}(x_{n_k}) - f_{n_k}(y_{n_k})| < 4\varepsilon, \quad \forall\ n_k \geq n_0$$

Taking $\varepsilon = \frac{\varepsilon_0}{4}$, we have a contradiction.

$\therefore$ $\mathcal{E}$ is equicontinuous. #

Application to Ordinary Differential Equations

Consider

$$(\text{IVP}) \begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

with f continuous (only, not necessary Lipschitz)

on $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$.

Of course, we cannot expect uniqueness result, but short time existence can be proved.

Idea of proof:

(Note: we only proved $\mathbb{R}^1$ case, but it still valid for high dims.)

(1) Weierstrass Approximation Theorem (on $\mathbb{R}^2$)

$\Rightarrow \exists \{p_n\}$ sequence of polynomials s.t.

$$d_\infty(p_n, f) \to 0 \quad (\text{in } C(R))$$

(2) Note that $\forall\, p_n$ satisfies Lipschitz condition (uniform in t). By Picard-Lindelöf Theorem

$\exists\, a_n' > 0$ with $a_n' < \min\{a, \frac{b}{M_n}, \frac{1}{L_n}\}$.

where $M_n = \|f_n\|_{\infty, R}$

$L_n =$ Lipschitz constant of f_n on R,

s.t. $\exists$ unique solution $x_n \in C^1[t_0 - a_n', t_0 + a_n']$

to the approximated (IVP)

$$\begin{cases} \dfrac{dx_n}{dt} = f_n(t, x_n) & \forall t \in [t_0 - a_n', t_0 + a_n'] . \\ x_n(t_0) = x_0 \end{cases}$$

(3) Then try to apply Ascoli's Theorem to $\{x_n\}$ and find a convergent subsequence

$x_{n_k} \to x$ for some function $x(t)$.

And hope that x is the required solution.

<u>Issue</u>: Since f is not assumed to satisfy the Lipschitz condition, one cannot expect $\{L_n\}$ is bounded (In fact, it is unbounded. Otherwise f satisfies Lipschitz condition.)

Then $\min\{a, \frac{b}{M_n}, \frac{1}{L_n}\} \to 0 \Rightarrow a_n' \to 0$.

We will not have an "interval" for the existence of the solution.

(On the other hand, as $p_n \to f$ in $(C(R), d_\infty)$, we have $M_n \le M$ for some $M > 0$.)

Therefore, to implement our plan, we need to improve the Picard-Lindelöf Theorem to

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<u>Prop 4.5</u> Under the setting of Picard-Lindelöf Theorem, $\exists$ unique solution $x(t)$ on the interval $[t_0 - a', t_0 + a']$ with $x(t) \in [x_0 - b, x_0 + b]$, where a' is any number satisfying

$$0 < a' < a^* = \min\{a, \frac{b}{M}\}.$$

Clearly, this implies $\exists$ unique solution on the open interval $(t_0 - a^*, t_0 + a^*)$.

Pf: Omitted (or postpone to the end of term if time allowed.)

Thm 4.6 (Cauchy-Peano Theorem)

Consider

$$(\text{IVP}) \quad \begin{cases} \frac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

where f is continuous on $R = [t_0-a, t_0+a] \times [x_0-b, x_0+b]$.

There exists $a' \in (0,a)$ and a C^1-function

$$x : [t_0-a', t_0+a'] \to [x_0-b, x_0+b]$$

solving the (IVP).

Pf: As in the "Idea of Proof",

$\exists$ sequence of polynomials $\{p_n\}$ s.t.

$$p_n \to f \quad \text{in } (C(R), d_\infty).$$

This implies $M_n = \|p_n\|_{\infty,R} \to M$, where $M = \|f\|_{\infty,R}$.

and p_n satisfies the Lipschitz condition.

(we don't need to worry about the Lip. constants)

By Prop 4.5, $\exists$ unique solution x_n defined on

$I_n = (t_0 - a_n, t + a_n)$, where $a_n = \min\{a, \frac{b}{M_n}\}$, for the (IVP) $\begin{cases} \frac{dx_n}{dt} = p_n(t, x_n) \\ x_n(t_0) = x_0 \end{cases}$, $t \in I_n$.

with $x_n(t) \in [x_0 - b, x_0 + b]$.

As $a_n = \min\{a, \frac{b}{M_n}\} \to \min\{a, \frac{b}{M}\} = a^*$,

for any fixed $a' < a^*$ $(a' > 0)$, $\exists n_0 > 0$ such that for $n \geq n_0$,

$$[t_0 - a', t_0 + a'] \subset I_n = (t_0 - a_n, t_0 + a_n).$$

Hence $\forall n \geq n_0$,

x_n is defined on $[t_0 - a', t_0 + a']$.

<u>Claim 1</u>: $\{x_n\} \subset C[t_0 - a', t_0 + a']$ is equicontinuous.

In fact, (IVP) $\Rightarrow$

$$\left|\frac{dx_n}{dt}\right| = |p_n(t, x_n)| \leq M_n \quad \forall t$$

Since $M_n \to M$, $\left\|\frac{dx_n}{dt}\right\|_\infty$ is uniformly bounded.

By Prop 4.1, $\{x_n\}$ is equicontinuous. ※

Claim 2: $\{x_n\}$ is bounded in $(C[t_0-a', t_0+a'], d_\infty)$

In fact, (IVP) $\Rightarrow$

$$x_n(t) = x_0 + \int_{t_0}^{t} f_n(s, x_n(s))\,ds, \quad \forall t \in [t_0-a', t_0+a']$$

$$\therefore \quad |x_n(t)| \le |x_0| + a' \sup_s |f_n(s, x_n(s))|$$

$$\le |x_0| + a' M_n$$

$\Rightarrow$ $\|x_n\|_{\infty, [t_0-a', t_0+a']}$ is uniformly bounded.

$\therefore$ $\{x_n\}$ is a bounded set in $(C[t_0-a', t_0+a'], d_\infty)$ ※

Then Claims 1 & 2 allow us to apply Ascoli's Theorem to conclude that $\exists$ a subsequence x_{n_j} in $(C[t_0-a', t_0+a'], d_\infty)$ uniformly converges to a cts. function x on $[t_0-a', t_0+a']$.

Claim 3: x solves (IVP) $\begin{cases} \frac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$.

Proof of Claim 3: We only need to show that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds.$$

Note that x_{n_j} satisfies

$$x_{n_j}(t) = x_0 + \int_{t_0}^{t} p_{n_j}(s, x_{n_j}(s))\,ds.$$

Clearly $x_{n_j}(t) \Rightarrow x(t)$ as $j \to +\infty$.

We only need to show that

$$\lim_{j\to\infty} \int_{t_0}^{t} p_{n_j}(s, x_{n_j}(s))\,ds = \int_{t_0}^{t} f(s, x(s))\,ds.$$

Since $f \in C(R)$ & R is closed & bounded in $\mathbb{R}^2$,

f is uniformly continuous on R.

Therefore, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that

$\forall\ (s_1, x_1), (s_2, x_2) \in R$ with

$$|s_1 - s_2| < \delta \text{ and } |x_1 - x_2| < \delta,$$

we have
$$|f(s_2, x_2) - f(s_1, x_1)| < \varepsilon.$$

On the other hand, $\|p_n - f\|_{\infty, R} \to 0$

$\Rightarrow \exists\, n_0 > 0$ s.t.

$$|p_n(s,x) - f(s,x)| < \varepsilon, \ \forall\, (s,x) \in R.$$

Therefore, for j sufficiently large such that

$$n_j \geq n_0 \quad \& \quad \|x_{n_j} - x\|_\infty < \delta,$$

we have

$$\left| \int_{t_0}^{t} f_{n_j}(s, x_{n_j}(s))ds - \int_{t_0}^{t} f(s, x(s))ds \right|$$

$$\leq \left| \int_{t_0}^{t} f_{n_j}(s, x_{n_j}(s))ds - \int_{t_0}^{t} f(s, x_{n_j}(s))ds \right|$$

$$+ \left| \int_{t_0}^{t} f(s, x_{n_j}(s))ds - \int_{t_0}^{t} f(s, x(s))ds \right|$$

$$\leq \int_{t_0}^{t} \left| f_{n_j}(s, x_{n_j}(s)) - f(s, x_{n_j}(s)) \right| ds$$

$$+ \int_{t_0}^{t} \left| f(s, x_{n_j}(s)) - f(s, x(s)) \right| ds$$

$$\leq \varepsilon \cdot a' + \varepsilon \cdot a' = 2\varepsilon a'.$$

This shows that $\int_{t_0}^{t} f_{n_j}(s, x_{n_j}(s))ds \to \int_{t_0}^{t} f(s, x(s))ds$

as $j \to +\infty$. #

Claim 3 completes the proof of the theorem. #