Proof of Picard-Lindelöf Theorem:

For $a'>0$ to be chosen later, we let

$$X = \{ \varphi \in C[t_0-a', t_0+a'] : \varphi(t_0) = x_0,\ \varphi(t) \in [x_0-b, x_0+b] \}$$

with (uniform) metric d_∞ on X.

First note that X is closed subset in the complete metric space $(C[t_0-a', t_0+a'], d_\infty)$. Hence (X, d_∞) is complete.

Define T on X by

$$(T\varphi)(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds$$

(This is well-defined as $\varphi(s) \in [x_0-b, x_0+b]$.)

Clearly $T\varphi \in C[t_0-a', t+a']$ & $(T\varphi)(t_0) = x_0$.

To show $T\varphi \in X$, we need

$$(T\varphi)(t) \in [x_0 - b, x_0 + b].$$

Let $M = \sup_{(t,x)\in R} |f(t,x)|$.

Then $\forall\, t \in [t_0 - a', t_0 + a']$,

$$\begin{aligned} |(T\varphi)(t) - x_0| &= \left| \int_{t_0}^{t} f(s, \varphi(s))\,ds \right| \\ &\leq M\,|t - t_0| \\ &\leq M a' \end{aligned}$$

If we choose $0 < a' \leq \frac{b}{M}$, then

$$|(T\varphi)(t) - x_0| \leq b$$

$$\Rightarrow \quad T\varphi \in X.$$

this is, for $0 < a' \leq \frac{b}{M}$,

$$T: X \to X \text{ is a self-map}$$

from a complete metric space (X, d_∞) to itself.

To see whether T is a contraction, we check

$$\left|(T\varphi_2 - T\varphi_1)(t)\right| = \left|\left(x_0 + \int_{t_0}^{t} f(s, \varphi_2(s))ds\right) - \left(x_0 + \int_{t_0}^{t} f(s, \varphi_1(s))ds\right)\right|$$

$$\leq \int_{t_0}^{t} \left|f(s, \varphi_2(s)) - f(s, \varphi_1(s))\right| ds$$

$$\leq L \int_{t_0}^{t} |\varphi_2(s) - \varphi_1(s)| ds \quad (\text{by Lip. condition})$$

$$\leq L |t - t_0| \sup_{[t_0 - a', t_0 + a']} |\varphi_2(s) - \varphi_1(s)|$$

$$\leq L a' d_\infty(\varphi_2, \varphi_1)$$

Therefore, if we further require $La' = \gamma < 1$,

then T is a contraction:

$$d_\infty(T\varphi_2, T\varphi_1) \leq \gamma d_\infty(\varphi_2, \varphi_1)$$

with $\gamma = La' < 1$.

In conclusion,

if $0 < a' < \min\{a, \frac{b}{M}, \frac{1}{L}\}$, then

$T: X \to X$ is a contraction on a complete metric space. Therefore, by Contraction Mapping Principle, T admits a <u>unique</u> fixed point $x(t) \in X$.

By Prop 3.11, we've proved Thm 3.10. $\#$

<u>Notes</u>:

(1) Existence part of Picard-Lindelöf Thm still holds with $f(t,x)$ cts only (without Lip. condition)

However, the solution may not be unique:

eg: Consider $f(t,x) = |x|^{1/2}$ on $\mathbb{R}\times\mathbb{R}$

f is cts, but not Lip. cts.

(Cauchy Problem) $\begin{cases} \frac{dx}{dt} = |x|^{1/2} & \text{on } \mathbb{R} \\ x(0) = 0 \end{cases}$

has solutions $x_1(t) \equiv 0 \quad \forall t \in \mathbb{R}$ and

$$x_2(t) = \begin{cases} \frac{1}{4}t^2, & t \ge 0 \\ -\frac{1}{4}t^2, & t < 0 \end{cases}$$

[Check: x_2 is differentiable with $\frac{dx_2}{dt} = \frac{1}{2}|t|, \ \forall t \in \mathbb{R}$
$= |x_2|^{1/2}$

& $x_2(0) = 0$.]

(2) Uniqueness holds regardless of the size of the interval of existence. (Proof omitted as it is more in the curriculum of ODE. See Prof Chou's notes for a proof.)

(3) The proof works for system of ODEs, just the x and f become vector-valued:

Thm 3.13 (Picard-Lindelöf Theorem for Systems)

Consider (IVP) $\begin{cases} \frac{dx}{dt} = f(t,x) \\ x(t_0) = x_0, \end{cases}$ $x_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

where $x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \in [x_1-b, x_1+b] \times \cdots \times [x_n-b, x_n+b]$ and

$$f(t,x) = \begin{pmatrix} f_1(t,x) \\ \vdots \\ f_n(t,x) \end{pmatrix} \in C^1(R), \text{ with}$$

$$R = [t_0-a, t_0+a] \times [x_1-b, x_1+b] \times \cdots \times [x_n-b, x_n+b],$$

satisfying (Lipschitz condition (uniform in t))

$$|f(t,x) - f(t,y)| \leq L|x-y|, \quad \forall (t,x), (t,y) \in R,$$

for some constant $L > 0$.

There exists a unique solution $x \in C^1[t_0-a', t_0+a']$ with

$$x(t) \in [x_1-b, x_1+b] \times \cdots \times [x_n-b, x_n+b], \quad \forall t \in [t_0-a', t_0+a']$$

to (IVP), where a' satisfies

$$0 < a' < \min\{a, \frac{b}{M}, \frac{1}{L}\}, \text{ here } M = \max_{j=1,\cdots,n} \sup_R |f_j(t,x)|.$$

(4) The Picard-Lindelöf Theorem for system can be applied to initial value problem for higher order ordinary differential equations:

$$
(\mathrm{IVP}) \begin{cases}
\dfrac{d^m x}{dt^m} = f\left(t, x, \dfrac{dx}{dt}, \cdots, \dfrac{d^{m-1}x}{dt^{m-1}}\right) \\
x(t_0) = x_0 \\
\dfrac{dx}{dt}(t_0) = x_1 \\
\vdots \\
\dfrac{d^{m-1}x}{dt^{m-1}}(t_0) = x_{m-1}
\end{cases}
$$

By letting $\vec{X} = \begin{pmatrix} x \\ \frac{dx}{dt} \\ \vdots \\ \frac{d^{m-1}x}{dt^{m-1}} \end{pmatrix}$, then

$$
\frac{d\vec{X}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{d^2x}{dt^2} \\ \vdots \\ \frac{d^m x}{dt^m} \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{d^2x}{dt^2} \\ \vdots \\ f\left(t, x, \frac{dx}{dt}, \cdots, \frac{d^{m-1}x}{dt^{m-1}}\right) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}
$$

$$
= \vec{f}(t, \vec{X})
$$

with $\vec{X}(t_0) = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix}$.

Ch 4 Space of Continuous Functions

§4.1 Ascoli's Theorem

Notation: If (X, d) = metric space, we denote

$$C_b(X) = \{ f \in C(X) : |f(x)| \le M, \forall x \in X, \text{ for some } M \}$$

the vector space of all bounded continuous functions on X. Clearly $C_b(X) \subset C(X)$.

($C(X)$ = set of continuous functions on X.)

eg: If G = (nonempty) bounded open set in $\mathbb{R}^n$, then

$$C_b(\bar{G}) = C(\bar{G})$$

as $\bar{G}$ is closed and bounded, $f \in C(\bar{G})$ has to be bounded.

Recall that: A norm $\|\cdot\|$ on a real vector space X is defined by the following properties:

(N1) $\|x\| \geq 0$ & "$\|x\| = 0 \Leftrightarrow x = 0$"

(N2) $\|\alpha x\| = |\alpha| \, \|x\|$ $\quad (\alpha \in \mathbb{R})$

(N3) $\|x+y\| \leq \|x\| + \|y\|$.

And a vector space with norm $(X, \|\cdot\|)$ is called a norm space.

A norm space has a natural metric

$$d(x,y) = \|x-y\|.$$

<u>Fact</u>: The <u>supnorm</u> $\|f\|_\infty = \sup_{x \in X} |f(x)|$

is a norm on $C_b(X)$.

And we always assume $C_b(X)$ with metric

$$d_\infty(f,g) = \|f-g\|_\infty.$$

given by the <u>supnorm</u>.

Similar to $((C[a,b], d_\infty)$, we have

$\boxed{\underline{\text{Prop}}: \ (C_b(X), d_\infty) \text{ is complete (for any metric space } (X,d))}$

$\underline{\text{Pf}}$: Let $\{f_n\}$ be a Cauchy seq. in $(C_b(X), d_\infty)$

Then $\forall \varepsilon > 0$, $\exists n_0 \geq 0$ s.t.

$$\|f_m - f_n\|_\infty < \frac{\varepsilon}{4}, \quad \forall m, n \geq n_0.$$

In particular, $\forall x \in X$,

$$(*)_1 \quad |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty < \frac{\varepsilon}{4}, \ \forall m, n \geq n_0$$

$\Rightarrow \{f_n(x)\}$ is a Cauchy seq. in $\mathbb{R}$.

By <u>completeness</u> of $\mathbb{R}$ (not X), $\lim_{n\to\infty} f_n(x)$ exists

and, in general, depends on x.

Let denote it by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \forall x \in X.$$

This gives a function f on X.

Claim (1). f is bounded.

Pf: Letting $m \to \infty$ in $(*)_1$, we have
$\forall \varepsilon > 0$, and $\forall x \in X$,

$$(*)_2 \qquad |f(x) - f_n(x)| \leq \frac{\varepsilon}{4}, \ \forall n \geq n_0$$

In particular, $|f(x) - f_{n_0}(x)| \leq \frac{\varepsilon}{4}, \forall \varepsilon > 0, \forall x \in X.$

$\Rightarrow \forall x \in X, \quad |f(x)| \leq \frac{\varepsilon}{4} + |f_{n_0}(x)| \leq \frac{\varepsilon}{4} + M_0,$

where M_0 is a bound for f_{n_0}.

$\therefore$ f is bounded.

Claim (2): f is continuous.

Pf: f_{n_0} cts $\Rightarrow \forall x_0 \in X$ & $\forall \varepsilon > 0, \exists \delta > 0$

s.t. $|f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{4}, \forall d(x, x_0) < \delta.$

Then together with $(*)_2$,

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)|$$

$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon, \quad \forall\, d(x, x_0) < \delta.$

$\therefore$ f is cts at x_0.

Since $x_0 \in X$ is arbitrary, f is cts on X.

Claims (1) & (2) $\Longrightarrow$ $f \in C_b(X)$.

Finally, by $(*)_2$, $\sup_{x \in X} |f(x) - f_n(x)| \leq \frac{\varepsilon}{4}, \forall n \geq n_0$

i.e. $d_\infty(f_n, f) \leq \frac{\varepsilon}{4}, \forall n \geq n_0$

So $d_\infty(f_n, f) \to 0$ as $n \to \infty$

That is $f_n \to f$ in $(C_b(X), d_\infty)$. #

Notes: (i) We've just proved that $(C_b(X), d_\infty)$ is a Banach space, i.e. a complete normed vector space.

(ii) $C_b(X)$ is usually of infinite dimensional:

egs: When $X = \mathbb{R}^n$ a subset with non-empty interia in $\mathbb{R}^n$.

Explicit eg: $X = [0,1] \subset \mathbb{R}$, then $\{x^n\}_{n=0}^{\infty} \subset C_b(X)$.

Clearly, $\{x^n\}_{n=0}^{\infty}$ is a linearly indep. subset.

$\Rightarrow$ $C_b(X) = C[0,1]$ is of infinite dimensional.

(iii) $C_b(X)$ could be of finite dimension:

eg: $X = \{p_1, \cdots, p_n\}$ finite set with discrete metric

Then
$$\begin{aligned} X &\to \mathbb{R}^n \\ f &\mapsto (f(p_1), \cdots, f(p_n)) \end{aligned}$$
is a linear bijection.

(iv) A reason for studying $C_b(X)$ instead of $C(X)$ is the fact that $C(X)$ may contain unbounded function and supnorm $\|\cdot\|_\infty$ doesn't define.

eg: $X = \mathbb{R} = (-\infty, +\infty)$.

However, in some cases, we still possible to define a metrix on $C(X)$.

eg $X = \mathbb{R}^n$, $\overline{B_n(0)} = \{|x| \le n\}$, $\forall n = 1, 2, 3, \cdots$

$\forall f \in C(\mathbb{R}^n)$, define

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{\infty, \overline{B_n(0)}}}{1 + \|f-g\|_{\infty, \overline{B_n(0)}}}.$$

where $\|\cdot\|_{\infty, \overline{B_n(0)}}$ is the supnorm on the closed ball $\overline{B_n(0)}$.

Then d is a complete metric on $C(\mathbb{R}^n)$.

(v) $C_b(X)$ may not have Bolzano-Weierstrass property.

Recall: Bolzano-Weierstrass Theorem (in $\mathbb{R}^n$):

every bounded sequence (set) has (contains) a convergent subsequence (sequence).

eg. $C_b([0,1]) = C[0,1]$. Let $f_n(x) = x^n$, $x \in [0,1]$.

Then $\|f_n\|_\infty = 1$, $\forall n$.

Note that pointwise limit $f_n(x) \to \begin{cases} 1, & x = 1 \\ 0, & \text{otherwise}. \end{cases}$

$\Rightarrow$ no subsequence converges in $C_b([0,1])$.

In view of note (V), we need further condition to help us to find convergence sequence in subset of $C_b(X)$.

Def: Let (X, d) be a metric space. A set $E \subset X$ is called a precompact set if every sequence in E contains a convergent subsequence (with limit in X, not necessary in E).

If further required that the limit belongs to E, then it is called compact.

Note: Compact set is a closed precompact set.

Pf: Let $\{x_n\} \subset E$.

E precompact $\Rightarrow \exists\ x_{n_j} \to z \in X$

E closed $\Rightarrow z \in E$

Hence closed precompact $\Rightarrow$ compact.

The other direction: "compact $\Rightarrow$ closed precompact" is trivial

#

eg: Bolzano-Weierstrass $\Rightarrow$

$E \subset \mathbb{R}^n$ is precompact $\Leftrightarrow$ E is bounded.

Hence $E \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ E is closed & bounded.

Def: Let (X, d) be a metric space. A subset $\mathcal{C}$ of $C(X)$ is <u>equicontinuous</u> if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \ \forall f \in \mathcal{C} \ \& \ d(x,y) < \delta \ (x, y \in X).$$

<u>Note</u>: Clearly if $\mathcal{C}$ is equicontinuous, then any $\mathcal{C}' \subset \mathcal{C}$ is equicontinuous.

<u>Eg</u>: If $X = \overline{G} \subset \mathbb{R}^n$, $G \neq \emptyset$ open & bounded. Then $f \in C(\overline{G})$ is always uniformly continuous:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon, \ \forall \ d_{\mathbb{R}^n}(x,y) = |x-y| < \delta \ (x, y \in \overline{G})$$

The δ here usually depends on f.

Comparing the definition of equicontinuity,

$\mathcal{C}$ is equicontinuous, if we can find a uniform $\delta > 0$ for all functions $f \in \mathcal{C}$,

i.e. δ is independent of points $x, y \in \bar{G}$ and functions $f \in \mathcal{C}$.

eg: A function f defined on a (non-empty open & bound G) subset $\bar{G}$ of $\mathbb{R}^n$ is called Hölder continuous if $\exists\ \alpha \in (0,1)$ such that

$$(*) \quad |f(x) - f(y)| \leq L|x-y|^{\alpha}, \quad \forall x, y \in \bar{G}$$

for some constant L.

The number α is called the Hölder exponent.

The function is called Lipschitz continuous if $(*)$ holds for $\alpha = 1$.

For a fixed $\alpha \in (0,1]$ & $L > 0$, the family

$$\mathcal{C} = \{ f \in C(\bar{G}) : f \text{ Hölder/Lip. with exponent } \alpha \text{ and } L > 0 \}$$

is an equicontinuous family.

Pf: $\forall \varepsilon>0$, let $\delta>0$ such that $L\delta^{\alpha}<\varepsilon$.

Then $\forall f \in \mathcal{E}$, $\forall x,y \in \bar{X}$ with $|x-y|<\delta$,

$$|f(x)-f(y)| \leq L|x-y|^{\alpha} < L\delta^{\alpha} < \varepsilon. \quad \#$$

Prop 4.1: Let $\mathcal{E}$ be a subset $C(\bar{G})$ where $\bar{G}$ is a (nonempty) convex in $\mathbb{R}^n$. Suppose that each function in (bounded open G) $\mathcal{E}$ is differentiable and there is a uniform bound on their partial derivatives.

Then $\mathcal{E}$ is equicontinuous.

(for same M)

$\left(\text{i.e. } \mathcal{E} = \left\{ f \in C(\bar{G}) : f \text{ differentiable}, \left\| \left|\frac{\partial f}{\partial x_i}\right| \right\|_{\infty} \leq M, \forall i \right\} \right.$ is equicontinuous provided $\bar{G}$ is convex.)

Pf: $\forall x,y \in \bar{G}$, $\bar{G}$ convex

$$\Rightarrow x+t(y-x) \in \bar{G}, \quad \forall t \in [0,1].$$

Then $f(y)-f(x) = \int_0^1 \frac{d}{dt} f(x+t(y-x))\,dt$

$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x+t(y-x))(y_i-x_i)\,dt$$

$$= \sum_{i=1}^n \left(\int_0^1 \frac{\partial f}{\partial x_i}(x+t(y-x))\,dt\right)(y_i-x_i)$$

$$\leq \sqrt{\sum_{i=1}^n \left|\int_0^1 \frac{\partial f}{\partial x_i}(x+t(y-x))\,dt\right|^2}\ |y-x|$$

$\leq \sqrt{n} M |y-x|$, where $M =$ unifom bd. on the partial derivatives.

Then by the above example, $\mathcal{C}$ is equicontinuous. ※