One can generalize eg 3.6 to

Prop 3.5: Let $\Phi(x) = x + \Psi(x) : U \to \mathbb{R}^n$ be C^1 on some open set $U \subset \mathbb{R}^n$ containing 0, such that

$$\Psi(0) = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{\partial \Psi_i}{\partial x_j}(x) = 0, \ \forall i, j.$$

Then $\exists\, r > 0$ and $R > 0$ such that $\forall y \in B_R(0)$,

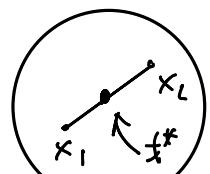
$\Phi(x) = y$ has a unique solution x in $B_r(0)$.

Pf: For $x_1, x_2 \in B_r(0)$ ($r > 0$ to be determined)

using remark(1)

consider $\varphi_i(t) = \Psi_i(x_1 + t(x_2 - x_1))$ for $t \in [0,1]$.

Then $\varphi_i(0) = \Psi_i(x_1)$, $\varphi_i(1) = \Psi_i(x_2)$.



$$\begin{aligned}
\varphi_i'(t) &= \frac{d}{dt} \Psi_i(x_1 + t(x_2 - x_1)) \\
&= \nabla \Psi_i(x_1 + t(x_2 - x_1)) \cdot (x_2 - x_1)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad |\Psi_i(x_2) - \Psi_i(x_1)| &= |\varphi_i(1) - \varphi_i(0)| \\
&= \left| \int_0^1 \varphi_i'(t)\, dt \right| \\
&\le \int_0^1 |\nabla \Psi_i(x_1 + t(x_2 - x_1)) \cdot (x_2 - x_1)|\, dt \\
&\le \left(\int_0^1 |\nabla \Psi_i(x_1 + t(x_2 - x_1))|\, dt \right) |x_2 - x_1|
\end{aligned}$$

$$\leq |\nabla \Psi_i(x_1 + t^*(x_2 - x_1))|\ |x_2 - x_1|$$

(for some $t^* \in (0,1)$ by Mean Value Thm, since Ψ is C^1)

Note that $x_1, x_2 \in B_r(0) \Rightarrow x_1 + t^*(x_2 - x_1) \in B_r(0)$

Let $M_r = \sup\limits_{x \in \overline{B_r}(0)} \left(\sum\limits_{i,j=1}^{n} \left| \frac{\partial \Psi_i}{\partial x_j}(x) \right|^2 \right)^{1/2} > 0$ (unless $\Psi \equiv 0$ in $\overline{B_r(0)}$ which is a trivial case.)

$$\Rightarrow |\Psi(x_2) - \Psi(x_1)| = \sqrt{\sum_{i=1}^{n} |\Psi_i(x_2) - \Psi_i(x_1)|^2}$$

$$\leq \sqrt{\sum_{i=1}^{n} |\nabla \Psi_i(x_1 + t^*(x_2 - x_1))|^2}\ |x_2 - x_1|$$

$$\leq M_r |x_2 - x_1|$$

By $\lim\limits_{x \to 0} \frac{\partial \Psi_i}{\partial x_j}(x) = 0$, $\forall i,j = 1, \cdots, n$, and Ψ is C^1,

$\exists\ r > 0$ s.t. $M_r \leq \frac{1}{2}$. $\Rightarrow |\Psi(x_2) - \Psi(x_1)| \leq \frac{1}{2}|x_2 - x_1|$.

Take $R = (1 - \frac{1}{2})r = \frac{r}{2}$. By Thm 3.4 & Remark (2),

$\forall\ y \in B_{\frac{r}{2}}(0)$, $\exists\ x \in B_r(0)$ s.t. $\Phi(x) = y$. #

eg 3.7: Let $g(x) \in C[0,1]$ and $K(x,t) \in C([0,1]\times[0,1])$.

Let $M = \|K\|_\infty = \max_{(x,t)\in[0,1]\times[0,1]} |K(x,t)|$.

Then $\forall\ g \in C[0,1]$ with $\|g\|_\infty < \frac{1}{8M}$,

$\exists$ unique solution $y \in C[0,1]$ with

$\|y\|_\infty \leq \frac{1}{4M}$

s.t.

$$\boxed{y(x) = g(x) + \int_0^1 K(x,t)\, y^2(t)\, dt}$$

(Integral Equation)

Pf: Note that $(C[0,1], \|\cdot\|_\infty)$ is a Banach space.

Consider $\Phi : \overline{B_r^\infty(0)} \to C[0,1]$ defined by ($r>0$ to be determined)

$$\begin{array}{ccc} \overline{B_r^\infty(0)} & \to & C[0,1] \\ \cup & & \Psi \\ y & \longmapsto & \Phi(y) \end{array}$$ s.t. $\forall x \in [0,1]$

$$\Phi(y)(x) = y(x) - \int_0^1 K(x,t)\, y^2(t)\, dt$$

Let $\Psi(y) : \overline{B_r^\infty(0)} \to C[0,1]$ be defined by

$$\Psi(y)(x) = -\int_0^1 K(x,t)\, y^2(t)\, dt.$$

Note also $\Phi(0) = 0$ (0 = zero function)

($\&$ $\Psi(0) = 0$)

$\forall\ y_1, y_2 \in \overline{B^\infty_r(0)}$,

$$\|\Phi(y_1) - \Phi(y_2)\|_\infty = \max_{x\in[0,1]} \left| -\int_0^1 K(x,t)y_1^2(t)dt + \int_0^1 K(x,t)y_2^2(t)dt \right|$$

$$\le \int_0^1 \left(\max_{x\in[0,1]} |K(x,t)|\right) |y_2^2(t) - y_1^2(t)|\, dt$$

$$\le M \|y_2^2 - y_1^2\|_\infty$$

$$\le M\ \|y_2+y_1\|_\infty \|y_2 - y_1\|_\infty$$

$$\le 2rM\ \|y_2 - y_1\|_\infty$$

Choose $r = \frac{1}{4M}$, then

$$\|\Phi(y_1) - \Phi(y_2)\|_\infty \le \frac{1}{2}\|y_1 - y_2\|_\infty, \quad \forall\ y_1, y_2 \in \overline{B^\infty_{\frac{1}{4M}}(0)}.$$

Hence Thm 3.4 $\Rightarrow$

$\forall\ g \in \overline{B^\infty_R(0)}$ with $R = (1-\frac{1}{2})r = \frac{1}{2}\cdot\frac{1}{4M} = \frac{1}{8M} > 0$,

$\exists!\ y \in \overline{B^\infty_{\frac{1}{4M}}(0)}$ s.t. $\Phi(y) = g$

i.e. $y(x) - \int_0^1 K(x,t)y^2(t)\,dt = g(x)$, $\forall\ x\in[0,1]$

which is the required solution to the integral equation. ※

§3.3 The Inverse Function Theorem

Recall: Chain Rule

Let $G: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $F: V \subset \mathbb{R}^m \longrightarrow \mathbb{R}^\ell$ $\Big\}$ differentiable,

U, V open in $\mathbb{R}^n$ & $\mathbb{R}^m$ respectively, and

$G(U) \subset V$.

Then $H = F \circ G : U \to \mathbb{R}^\ell$ differentiable and

$$DH(x) = DF(G(x))\, DG(x),$$

where
$$DG(x) = \left(\frac{\partial G_i}{\partial x_j}(x)\right)_{\substack{i=1,\dots,m\\ j=1,\dots,n}} = \begin{pmatrix} - \nabla G_1 - \\ \vdots \\ - \nabla G_m - \end{pmatrix} = \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial G_m}{\partial x_1} & \cdots & \frac{\partial G_m}{\partial x_n} \end{pmatrix}$$

and similarly for DF & DH.

We also need

Prop 3.6 Let $F: B \to \mathbb{R}^n$ be C^1, where B = ball in $\mathbb{R}^n$.

Then $\forall\, x_1, x_2 \in B$,

matrix acts on column vector.

$$F(x_1) - F(x_2) = \left(\int_0^1 DF(x_2 + t(x_1 - x_2))\, dt\right) \cdot (x_1 - x_2)$$

In component form $F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$, this is

$$F_i(x_1) - F_i(x_2) = \sum_{j=1}^n \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(x_2 + t(x_1 - x_2))\, dt\right)(x_1 - x_2)_j.$$

Pf: For each $i = 1, \cdots, n$,

$$F_i(x_1) - F_i(x_2) = \int_0^1 \left(\frac{d}{dt} F_i(x_2 + t(x_1 - x_2)) \right) dt$$

$$= \int_0^1 \sum_{j=1}^{n} \left[\frac{\partial F_i}{\partial x_j}(x_2 + t(x_1 - x_2)) \cdot (x_1 - x_2)_j \right] dt$$

$$= \int_0^1 \nabla F_i(x_2 + t(x_1 - x_2)) \cdot (x_1 - x_2)\, dt$$

$$= \left(\int_0^1 \nabla F_i(x_2 + t(x_1 - x_2)) dt \right) \cdot (x_1 - x_2)$$

dot product of vectors

$$\therefore \quad F(x_1) - F(x_2) = \left(\int_0^1 DF(x_2 + t(x_1 - x_2)) dt \right) \cdot (x_1 - x_2)$$

#

Recall: If $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at a point p in an open set U of $\mathbb{R}^n$,

Then

$$F(p + x) - F(p) = DF(p)x + o(|x|)$$

$$\forall \; x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ sufficiently small,}$$

(i.e. $|x|$ small)

Where $o(|x|)$ is a remaining term such that

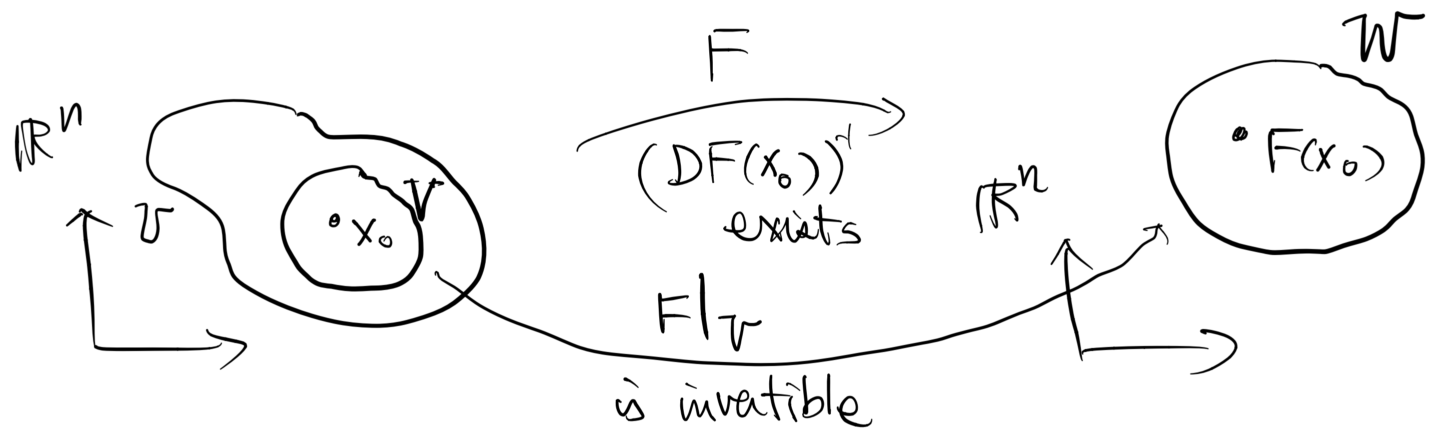
$$\frac{o(|x|)}{|x|} \to 0 \quad \text{as} \quad |x| \to 0.$$

Thm 3.7 (Inverse Function Theorem)

Let $F : U \to \mathbb{R}^n$ be a C^1-map from an open set $U \subset \mathbb{R}^n$. Suppose $x_0 \in U$ and $DF(x_0)$ is invertible (as a matrix or linear transformation).

(a) Then $\exists$ open sets V & W containing x_0 and $F(x_0)$ respectively such that the restriction of F on V is a bijection onto W with a C^1-inverse.

(b) The inverse is C^k when F is C^k, $1 \le k \le \infty$, in V.



Note: We only have local invertibility by the IFT.