eg3.5 Let $f:[0,1]\to[0,1]$ continuously differentiable with $|f'(x)|<1$ on $[0,1]$. Then f has a fixed point in $[0,1]$.

Pf: By mean value theorem

$\forall\, x,y\in[0,1]$, $\exists\, z\in[0,1]$ s.t.

$$f(x)-f(y)=f'(z)(x-y)$$

$$\Rightarrow \quad |f(x)-f(y)|\le|f'(z)|\,|x-y|$$

$$\le\Big(\sup_{[0,1]}|f'(z)|\Big)\,|x-y|.$$

Since $|f'(z)|<1$ & $f'(z)$ cts on $[0,1]$,

$$\gamma=\sup_{[0,1]}|f'(z)|\in[0,1).$$

If $\gamma=0$, then $f\equiv c$ on $[0,1]$ $\Rightarrow$ $f(c)=c$.

If $\gamma\neq0$, then $\gamma\in(0,1)$ & $|f(x)-f(y)|\le\gamma|x-y|$ $\forall\, x,y\in[0,1]$.

$\Rightarrow$ f is a contraction on the <u>complete</u> metric space $([0,1],\text{standard})$.

By contraction mapping principle, f has a fixed point. #

Def: If a normed space $(\mathbb{X}, \|\cdot\|)$ is complete as a metric space with respect to the induced metric $d(x,y) = \|x-y\|$, $\forall x, y \in \mathbb{X}$. Then it is called a <u>Banach space</u>.

e.g. - $(\mathbb{R}^n, \|\cdot\|_p)$ $(p > 1)$ is a Banach space.

- $(C[a,b], \|\cdot\|_\infty)$ is a Banach space.

Thm 3.4 (<u>Perturbation of Identity</u>)

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space, and

$$\Phi : \overline{B_r(x_0)} \to \mathbb{X} \text{ satisfies } \Phi(x_0) = y_0.$$

Suppose that $\Phi = Id_{\mathbb{X}} + \Psi$ such that

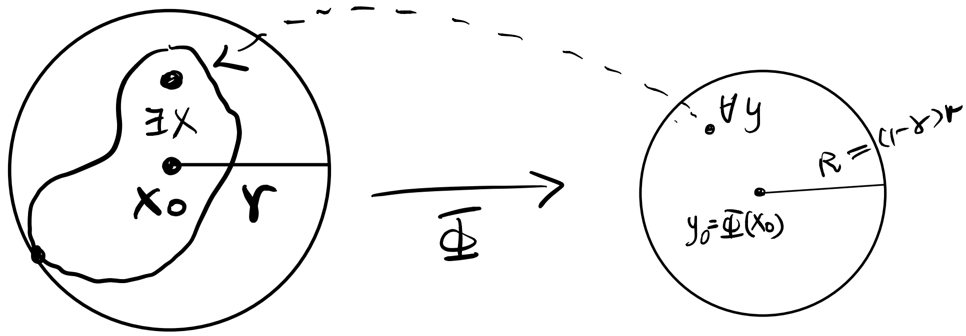
$\exists$ constant $\gamma \in (0,1)$ such that

$$\|\Psi(x_2) - \Psi(x_1)\| \le \gamma \|x_2 - x_1\|, \ \forall x_1, x_2 \in \overline{B_r(x_0)}.$$

Then $\forall y \in \overline{B_R(y_0)}$, where $R = (1-\gamma) r$,

$\exists$ unique $x \in \overline{B_r(x_0)}$ such that

$$\Phi(x) = y.$$



(i.e. Φ is locally invertible)

Idea of proof: $y = \Phi(x) = (\mathrm{Id}_{\mathbb{X}} + \Psi)(x) = x + \Psi(x)$

$\Leftrightarrow \quad x = y - \Psi(x)$

If $\forall y \in \overline{B_R(x_0)}$, define $Tx = y - \Psi(x)$.

Then $y = \Phi(x) \Leftrightarrow Tx = x$ (i.e. x is a fixed point of T).

<u>Proof</u>: Define $\tilde{\Phi} : \overline{B_r(0)} \to \mathbb{X}$ by

$$\begin{aligned}\tilde{\Phi}(x) &= \Phi(x+x_0) - \Phi(x_0) \\ &= \left(x + x_0 + \Psi(x+x_0)\right) - \left(x_0 + \Psi(x_0)\right) \\ &= x + \left[\Psi(x+x_0) - \Psi(x_0)\right] = x + \tilde{\Psi}(x)\end{aligned}$$

Then $\tilde{\Phi}(0) = 0$.

Further define, for any $y \in \overline{B_R(0)}$ $\quad (R = (1-\gamma)r)$

the map $T : \overline{B_r(0)} \to \mathbb{X}$ by

$$Tx = y - \left(\Psi(x+x_0) - \Psi(x_0)\right) \quad \left(\begin{aligned} &= y - \tilde{\Psi}(x) \\ &= x - \left(\tilde{\Phi}(x) - y\right)\end{aligned}\right)$$

Then $\forall x \in \overline{B_r(0)}$,

$$\|Tx\| = \|y - (\Psi(x+x_0) - \Psi(x_0))\|$$

$$\leq \|y\| + \|\Psi(x+x_0) - \Psi(x_0)\|$$

$$\leq \|y\| + \gamma\|x\|$$

$$\leq R + \gamma r$$

$$= (1-\gamma)r + \gamma r = r$$

$\therefore \quad T: \overline{B_r(0)} \to \overline{B_r(0)}$

And $\forall x_1, x_2 \in \overline{B_r(0)}$

$$\|Tx_1 - Tx_2\| = \|[y - (\Psi(x_1+x_0) - \Psi(x_0))] - [y - (\Psi(x_2+x_0) - \Psi(x_0))]\|$$

$$= \|\Psi(x_1+x_0) - \Psi(x_2+x_0)\|$$

$$\leq \gamma\|x_1 - x_2\|$$

Since $\gamma \in (0,1)$, $T: \overline{B_r(0)} \to \overline{B_r(0)}$ is a contraction.

Since $\overline{B_r(0)}$ is a closed subset and $(X, \|\cdot\|)$ is complete, Prop 3.1 $\Rightarrow$ $\overline{B_r(0)}$ is also complete.

Hence one can apply Contraction Mapping Principle to conclude that $\exists$ unique $x \in \overline{B_r(0)}$ s.t.

$$Tx = x \quad \text{in} \quad \overline{B_r(0)}.$$

i.e.

$$\begin{aligned} x &= y - \left(\Psi(x+x_0) - \Psi(x_0) \right) \\ &= y - \left[\left(\Phi(x+x_0) - (x+x_0) \right) - \left(\Phi(x_0) - x_0 \right) \right] \\ &= y - \Phi(x+x_0) + \Phi(x_0) + x \end{aligned}$$

$$\Leftrightarrow \quad \Phi(x+x_0) = y + y_0 \qquad (y_0 = \Phi(x_0))$$

Note that $y + y_0 \in \overline{B_R(y_0)}$ is arbitrary, and $x + x_0 \in \overline{B_r(x_0)}$, we've proved the Thm. #

Remarks

(1) Only need to assume $\|\Psi(x_1) - \Psi(x_2)\| \le \gamma \|x_1 - x_2\|$, $\gamma \in (0,1)$ for $x_1, x_2 \in B_r(x_0)$ (open ball). Then it is easy to get the same inequality for all $x_1, x_2 \in \overline{B_r(x_0)}$.

(2) Actually one can prove more that if $y \in B_R(y_0)$ (open ball), then the solution $x \in B_r(x_0)$ (open ball). (check the details of the pf.)

(3) The Thm $\Rightarrow$ $\Phi^{-1}: \overline{B_R(y_0)} \to \overline{B_r(x_0)}$ exists.

Claim $\|\Phi^{-1}(y_1) - \Phi^{-1}(y_2)\| \le \frac{1}{1-\gamma}\|y_1 - y_2\|$, $\forall y_1, y_2 \in \overline{B_R(y_0)}$.

In particular, Φ^{-1} is uniformly continuous (in fact "Lipschitz").

Pf: Let $x_i = \Phi^{-1}(y_i)$. Then x_i is the fixed point such that $x_i = y_i - \Psi(x_i)$.

$$\Rightarrow \|\Phi^{-1}(y_1) - \Phi^{-1}(y_2)\| = \|(y_1 - \Psi(x_1)) - (y_2 - \Psi(x_2))\|$$

$$\begin{aligned} &\le \|y_1 - y_2\| + \|\Psi(x_1) - \Psi(x_2)\| \\ &\le \|y_1 - y_2\| + \gamma\|x_1 - x_2\| \\ &= \|y_1 - y_2\| + \gamma\|\Phi^{-1}(y_1) - \Phi^{-1}(y_2)\| \end{aligned}$$

$$\Rightarrow \|\Phi^{-1}(y_1) - \Phi^{-1}(y_2)\| \le \frac{1}{1-\gamma}\|y_1 - y_2\| \quad \#$$

eg 3.6: $3x^4 - x^2 + x = \underbrace{-0.05}_{\text{small}}$ has a real root.

(Observation: and $3x^4 - x^2 + x = 0$ has a root $x = 0$.
Idea: look for solution near $x=0$ using Thm 3.4.)

Pf: Let
$$\begin{aligned} \Phi(x) &= x + (3x^4 - x^2) \\ &= x + \Psi(x), \end{aligned}$$

where $\Psi(x) = 3x^4 - x^2$ ("small" near $x=0$)

Then $\Phi(0)=0$.

And for $x_1, x_2 \in \overline{B_r(0)}$ ($r>0$ to be determined)

$$
\begin{aligned}
|\Phi(x_1)-\Phi(x_2)| &= |(3x_1^4-x_1^2)-(3x_2^4-x_2^2)| \\
&= |3(x_1^4-x_2^4)-(x_1^2-x_2^2)| \\
&= |3(x_1^3+x_1^2x_2+x_1x_2^2+x_2^3)-(x_1+x_2)||x_1-x_2| \\
&\le (12r^3+2r)|x_2-x_1|.
\end{aligned}
$$

Hence, we need to choose $r>0$ small enough

such that $\gamma = 12r^3+2r<1$

Also, in order to include $-0.05 \in \overline{B_R(0)}$,

we need $R=(1-\gamma)r \ge 0.05$.

Prof Chow provides a choice $r=\frac{1}{4}$.

Then $\gamma=\frac{11}{16}<1$ and $R=(1-\gamma)r=\frac{5}{64} \sim 0.078$.

By Thm 3.4, $\forall y \in \overline{B_{\frac{5}{64}}(0)}$, $\exists x \in \overline{B_{\frac{1}{4}}(0)}$ s.t.

$$\Phi(x)=y.$$

i.e. $$x+3x^4-x^2=y.$$

In particular, $-0.05 \in \overline{B_{\frac{5}{64}}(0)}$, we has a root of

$$x+3x^4-x^2=-0.05.$$

#