Q.1

Let $f_n(x) := \frac{x^n}{n}$ for $x \in [0,1]$. Show that the sequence (f_n) of differentiable functions converges uniformly to a differentiable function f on $[0,1]$, and that the sequence (f_n') converges on $[0,1]$ to a function g, but that $g(1) \neq f'(1)$.

Solution:

For each $x \in [0,1]$, $f_n(x) = \frac{x^n}{n} \to 0$ as $n \to \infty$

Let $f(x) = 0$, $\forall x \in [0,1]$. It is clear that f is differentiable.

$\therefore$ $f_n \to f$ pointwisely

Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ so that $\frac{1}{n} < \varepsilon$ $\forall n \geq N$. (Archimedean property)

Then $\forall x \in [0,1]$, when $n \geq N$, we have

$$|f_n(x) - f(x)| = \left|\frac{x^n}{n}\right| \leq \frac{1}{n} < \varepsilon$$

$\therefore$ $f_n \to f$ uniformly.

$f_n'(x) = x^{n-1}$, $\forall x \in [0,1]$.

For $x \in [0,1)$, $f_n'(x) \to 0$ as $n \to \infty$.

For $x = 1$, $f_n'(1) = 1 \to 1$ as $n \to \infty$.

Let $g(x) = \begin{cases} 0 & \text{for } x \in [0,1) \\ 1 & \text{for } x = 1 \end{cases}$.

Then $f_n' \to g$ pointwisely.

$g(1) = 1 \neq 0 = f'(1)$

Q.2

Let $I := [a,b]$ and let (f_n) be a sequence of functions on $I \to \mathbb{R}$ that converges on I to f. Suppose that each derivative f_n' is continuous on I and that the sequence (f_n') is uniformly convergent to g on I. Prove that

$$f(x) - f(a) = \int_a^x g(t)\,dt \quad \text{and that} \quad f'(x) = g(x) \text{ for all } x \in I.$$

Solution:

Note that by the FTC,

$$f_n(x) - f_n(a) = \int_a^x f_n'(t)\,dt \;, \forall\, x \in I. \quad -(*)$$

We want to take limits at both sides to yield the identities required.

For each $x \in I$, $\lim_{n\to\infty} (f_n(x) - f_n(a)) = f(x) - f(a)$ by pointwise convergence.

We want to show that $\lim_{n\to\infty} \int_a^x f_n'(t)\,dt = \int_a^x g(t)\,dt$

Since $f_n' \to g$ uniformly, $\forall\, t \in [a,b]$, $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t.

$$|f_n'(t) - g(t)| < \frac{\varepsilon}{b-a}.$$

$$\left| \int_a^x f_n'(t)\,dt - \int_a^x g(t)\,dt \right| = \left| \int_a^x (f_n'(t) - g(t))\,dt \right|$$

$$\le \int_a^x |f_n'(t) - g(t)|\,dt$$

$$< \int_a^x \varepsilon\,dt$$

$$= \frac{\varepsilon}{b-a}(x-a)$$

$$\le \frac{\varepsilon}{b-a} \cdot (b-a)$$

$$= \varepsilon \qquad \text{as} \quad n \ge N$$

$$\therefore \lim_{n\to\infty} \int_a^x f_n'(t)\,dt = \int_a^x g(t)\,dt$$

Then by letting $n \to \infty$ at $(*)$, we have.

$$f(x) - f(a) = \int_a^x g(t)\,dt \quad \forall\, x \in I.$$

Since f_n' are continuous, g is cont. by unif. conv.

Differentiating both side w.r.t. x, $f'(x) = g(x)$ by FTC.

Q.3

Let $\{r_1, r_2, \ldots, r_n, \ldots\}$ be an enumeration of rational numbers in $I := [0,1]$, and let $f_n : I \to \mathbb{R}$ be defined to be 1 if $x = r_1, \ldots, r_n$ and equal to 0 otherwise. Show that f_n is Riemann integrable for each $n \in \mathbb{N}$, that

$f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x) \leq \ldots$ and that $f(x) := \lim_{n\to\infty}(f_n(x))$ is the Dirichlet function, which is not Riemann integrable on $[0,1]$.

Solution:

For each n, note that the set $\{r_1, \ldots, r_n\}$ is a null set.

For each $x \in I \setminus \{r_1, \ldots, r_n\}$, let $\varepsilon_x = \frac{\min\{|x-r_1|, \ldots, |x-r_n|\}}{2}$ and I_x be the ε_x-nbd of x.

Then $f_n(y) = 0 \quad \forall y \in I_x \Rightarrow \lim_{y\to x} f_n(y) = 0 = f_n(x)$.

$\therefore f_n$ is continuous on $I \setminus \{r_1, \ldots, r_n\}$.

$\therefore f_n$ is Riemann integrable by the Lebesgue Criterion.

For each $x \in [0,1] \cap \mathbb{Q}^c$, note that $f_n(x) = 0 \ \forall n$

$\therefore (f_n(x))$ is increasing and $f(x) = \lim_{n\to\infty} f_n(x) = 0$

For each $x \in [0,1] \cap \mathbb{Q}$, $x = r_k$ for some $k \in \mathbb{N}$.

$\therefore f_n(x) = 1$ for $n \geq k$.

$\therefore (f_n(x))$ is increasing and $f(x) = \lim_{n\to\infty} f_n(x) = 1$.

$\therefore \ f(x) = \begin{cases} 1 & \text{for } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$, which is the Dirichlet function.

f is not Riemann integrable by the same proof of Q.3 in tutorial 7.