Q.1

Use the following argument to prove the Substitution Theorem 7.3.8.

Recall Theorem 7.3.8:

Let $J=[\alpha,\beta]$ and let $\varphi: J \to \mathbb{R}$ have a continuous derivative on J.

If $f: I \to \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$, then

$$\int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

Define $F(u) := \int_{\varphi(\alpha)}^{u} f(x)\,dx$ for $u \in I$, and $H(t) := F(\varphi(t))$ for $t \in J$. Show that $H'(t) = f(\varphi(t))\,\varphi'(t)$ for $t \in J$ and that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = F(\varphi(\beta)) = H(\beta) = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

Solution:

Since f is continuous on I, by the Fundamental Theorem of Calculus (FTC), F is differentiable and $F'(u) = f(u)$.

Therefore, by chain rule, $H'(t) = F'(\varphi(t))\,\varphi'(t) = f(\varphi(t))\,\varphi'(t)$ for $t \in J$

Let $\hat{H}(t) = \int_\alpha^t f(\varphi(s))\,\varphi'(s)\,dt$, $t \in J$. Since f, φ' are continuous, by FTC again, $\hat{H}'(t) = f(\varphi(t))\,\varphi'(t)$.

$\therefore \quad H'(t) = \hat{H}'(t) \quad \forall t \in J$

$(H(t) - \hat{H}(t))' = 0$.

Then $\forall t \in J$, $H(t) - \hat{H}(t) = C$ for some constant $C \in \mathbb{R}$

Take $t=\alpha$. $C = H(\alpha) - \hat{H}(\alpha) = F(\varphi(\alpha)) - \int_\alpha^\alpha f(\varphi(s))\,\varphi'(s)\,dt \xrightarrow{} 0$

$$= \int_{\varphi(\alpha)}^{\varphi(\alpha)} f(x)\,dx = 0$$

$\therefore\ H(t) - \hat{H}(t) = 0 \quad \forall t \in J$

$$\int_{\varphi(\alpha)}^{\varphi(t)} f(x)\,dx = \int_\alpha^t f(\varphi(s))\,\varphi'(s)\,ds$$

Take $t=\beta$

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_\alpha^\beta f(\varphi(s))\,\varphi'(s)\,ds.$$

Q.2

Let $f, g \in R[a,b]$.

(a) If $t \in \mathbb{R}$, show that $\int_a^b (tf \pm g)^2 \geq 0$.

(b) Use (a) to show that $2\left|\int_a^b fg\right| \leq t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2$ for $t>0$

(c) If $\int_a^b f^2 = 0$, show that $\int_a^b fg = 0$.

(d) Now prove that $\left|\int_a^b fg\right|^2 \leq \left(\int_a^b |fg|\right)^2 \leq \left(\int_a^b f^2\right)\cdot\left(\int_a^b g^2\right)$.

Solution:

(a) Clearly, $(tf \pm g)^2 \geq 0$.

<u>Claim</u>: If $F \in R[a,b]$ such that $F(x) \geq 0$ for $x \in [a,b]$, then

$$\int_a^b F(x)\,dx \geq 0$$

<u>Proof</u>:

Let $\dot{P}_n$ be any tagged partition of $[a,b]$ with $\|\dot{P}_n\| \to 0$.

Since $F(x) \geq 0$ on $[a,b]$, $S(f;\dot{P}_n) \geq 0$.

Recall by tutorial 5 Q.1, $\lim_{n\to\infty} S(f;P_n) = \int_a^b F(x)\,dx$.

Then $\int_a^b F(x)\,dx \geq 0$. //

Then we apply the claim to $(tf \pm g)^2$ to obtain the desired result.

(b) $\int_a^b (tf+g)^2 = \int_a^b (t^2f^2+2tfg+g^2) \geq 0$

$$-2t\int_a^b fg \leq t^2\int_a^b f^2 + \int_a^b g^2$$

$$-2\int_a^b fg \leq t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2$$

$\int_a^b (tf-g)^2 = \int_a^b (t^2f^2-2tfg+g^2) \geq 0$

$$2t\int_a^b fg \leq t^2\int_a^b f^2 + \int_a^b g^2$$

$$2\int_a^b fg \leq t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2$$

$\therefore \quad 2\left|\int_a^b fg\right| \leq t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2$

(c) If $\int_a^b f^2 = 0$,

$2\left|\int_a^b fg\right| \leq \frac{1}{t}\int_a^b g^2$

Note that $\lim_{t\to\infty} \frac{1}{t}\int_a^b g^2 = 0$.

By Sandwich Theorem, $\int_a^b fg = 0$.

(d) For the first inequality, it suffices to show that

$$\left|\int_a^b fg\right| \leq \int_a^b |fg|.$$

Note that $|fg| \pm fg \geq 0$. By the Claim in (a),

$$\int_a^b (|fg| \pm fg) \geq 0$$

$$\pm\int_a^b fg \leq \int_a^b |fg|$$

$$\Rightarrow \left|\int_a^b fg\right| \leq \int_a^b |fg|$$

For the second inequality, we consider

$$\int_a^b (tf-g)^2 \geq 0$$

$$\left(\int_a^b f^2\right) t^2 - \left(2\int_a^b fg\right) t + \int_a^b g^2 \geq 0 \qquad (*)$$

The LHS can be regarded as a quadratic function of t.

Condition (*) implies that this quadratic function has at most one real root. Therefore, by considering the discriminant, we get

$$\left(-2\int_a^b fg\right)^2 - 4\left(\int_a^b f^2\right)\left(\int_a^b g^2\right) \leq 0$$

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$$

By replacing f, g by $|f|, |g|$ respectively, we obtain the second inequality.

Q.3

Let $h:[0,1] \to \mathbb{R}$ be the Thomae's function and let sgn be the signum function. Show that the composite $\text{sgn} \circ h$ is not Riemann integrable on $[0,1]$.

Solution:

Recall the definition of Thomae's function and the signum function.

$$h(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \cap (0,1] \text{ and } x = \frac{p}{q}, p, q \in \mathbb{Z} \text{ s.t. } \gcd(p,q) = 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then we have

$$(\text{sgn} \circ h)(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\text{sgn} \circ h$ is bounded. We want to show that $\text{sgn} \circ h$ is not continuous everywhere on $[0,1]$. This shows that $\text{sgn} \circ h$ is not Riemann integrable by the Lebesgue's Integrability Criterion.

Let $f = \text{sgn} \circ h$.

<u>Case 1</u>: $a \in \mathbb{Q}$.

Then $f(a) = 1$. Let $\varepsilon = \frac{1}{2}$.

By density of $\mathbb{Q}^c$ on $\mathbb{R}$, $\forall \delta > 0, \exists x \in (a-\delta, a+\delta) \cap [0,1] \cap \mathbb{Q}^c$. $\Rightarrow f(x) = 0$

$\Rightarrow |f(x) - f(a)| = 1 > \frac{1}{2}$.

$\therefore f$ is not continuous at a.

<u>Case 2</u>: $a \in \mathbb{Q}^c$

Then $f(a) = 0$. Let $\varepsilon = \frac{1}{2}$.

By density of $\mathbb{Q}$ on $\mathbb{R}$, $\forall \delta > 0, \exists x \in (a-\delta, a+\delta) \cap [0,1] \cap \mathbb{Q}$. $\Rightarrow f(x) = 1$.

$\Rightarrow |f(x) - f(a)| = 1 > \frac{1}{2}$.

$\therefore f$ is not continuous at a.