§9.2 Tests for Absolute Convergence

Thm 9.2.1 (Limit Comparison Test II)

Suppose
- $x_n, y_n \neq 0$, $\forall n = 1, 2, \cdots$
- $\lim_{n\to\infty} \left| \frac{x_n}{y_n} \right| = r$ exists

Then

(a) If $r \neq 0$, then

$\sum x_n$ absolutely convergent $\Leftrightarrow$ $\sum y_n$ absolutely convergent

(b) If $r = 0$ and $\sum y_n$ absolutely convergent, then $\sum x_n$ is absolutely convergent

(only $\sum y_n \Rightarrow \sum x_n$, $\not\Leftarrow$ in this case)

Pf: Recall Limit Comparison Test (Thm 3.7.8) that

$$x_n, y_n > 0, \quad r = \lim_{n\to\infty} \frac{x_n}{y_n} \text{ exists}$$

then
- If $r \neq 0$, $\sum x_n$ converges $\Leftrightarrow$ $\sum y_n$ converges
- If $r = 0$, $\sum y_n$ converges $\Rightarrow$ $\sum x_n$ converges.

Applying Thm 3.7.8 to $\sum |x_n|$ & $\sum |y_n|$ $\#$

Recall also Comparison Test (Thm 3.7.7): $0 \le x_n < y_n$, $\forall n \ge K$ (for some $K \in \mathbb{N}$)

then
(a) $\sum y_n$ converges $\Rightarrow$ $\sum x_n$ converges
(b) $\sum x_n$ diverges $\Rightarrow$ $\sum y_n$ diverges.

Thm 9.2.2 (Root Test) (Cauchy)

(a) If $\exists\, r<1$ and $K \in \mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \le r, \quad \forall\, n \ge K,$$

then $\sum x_n$ is absolutely convergent.

(b) If $\exists\, K \in \mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \ge 1, \quad \forall\, n \ge K,$$

then $\sum x_n$ is divergent.

Pf: (a) If $|x_n|^{\frac{1}{n}} \le r$, $\forall\, n \ge K$

then $|x_n| \le r^n$, $\forall\, n \ge K$

Since $\sum r^n$ is convergent for $0 \le r < 1$,

Comparison Test 3.7.7 $\Rightarrow$ $\sum |x_n|$ is convergent.

(b) If $|x_n|^{\frac{1}{n}} \ge 1$, then $|x_n| \ge 1$, $\forall\, n \ge K$

$\Rightarrow$ $x_n \not\to 0$ as $n \to \infty$

$\Rightarrow$ $\sum x_n$ is divergent (n^{th} Term Test 3.7.3)

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Cor 9.2.3 Suppose $r = \lim_{n\to\infty} |x_n|^{\frac{1}{n}}$ exists.

Then
- $r < 1 \Rightarrow \sum x_n$ is absolutely convergent
- $r > 1 \Rightarrow \sum x_n$ is divergent.

(No conclusion for $r = 1$. see Eg 9.2.7 (b) later)

Pf: If $r<1$, then $\forall\ r<r_1<1$, $\exists\ K\in\mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \leq r_1 < 1, \quad \forall\ n\geq K,$$

then part (a) of Root Test $\Rightarrow$ $\sum x_n$ absolutely convergent.

If $r>1$, then $\exists\ K\in\mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} > 1, \quad \forall\ n\geq K,$$

then part (b) of Root Test $\Rightarrow$ $\sum x_n$ divergent. #

Thm 9.2.4 (Ratio Test) (D'Alembert)

Let $x_n \neq 0$, $\forall\ n=1,2,3,\cdots$

(a) If $\exists\ 0<r<1$ and $K\in\mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \leq r, \quad \forall\ n\geq K,$$

then $\sum x_n$ is absolutely convergent

(b) If $\exists\ K\in\mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1, \quad \forall\ n\geq K,$$

then $\sum x_n$ is divergent.

Pf: (a) $\forall\ n\geq K$, $|x_n| \leq r|x_{n-1}| \leq r^2|x_{n-2}| \leq \cdots \leq r^{n-K}|x_K|$

If $0<r<1$, then $\sum y_n = \sum r^{n-K}|x_K| = \frac{|x_K|}{r^K}\sum r^n$ is convergent

Comparison Test 3.7.7 $\Rightarrow$ $\sum |x_n|$ is convergent.

i.e. $\sum x_n$ is absolutely convergent.

(b) $\forall n \geq K$, $|x_n| \geq |x_{n-1}| \geq |x_{n-2}| \geq \cdots \geq |x_K|$

$\therefore$ $x_n \not\to 0$ as $n \to \infty$ $\Rightarrow$ $\sum x_n$ is divergent. #

Cor 9.2.5 If • $x_n \neq 0$, $\forall n = 1, 2, 3, \cdots$, and

• $r = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$ exists

Then
- • $r < 1 \Rightarrow \sum x_n$ is absolutely convergent.
- • $r > 1 \Rightarrow \sum x_n$ is divergent

(No conclusion for $r = 1$. see Eg 9.2.7(c) later)

Pf: If $r < 1$, then $\forall r_1 \in (r, 1)$, $\exists K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| < r_1 < 1 \quad , \forall n \geq K$$

Part (a) of Thm 9.2.4 $\Rightarrow$ $\sum x_n$ is absolutely convergent.

If $r > 1$, then $\exists K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| > 1, \ \forall n \geq K$$

Part (b) of Thm 9.2.4 $\Rightarrow$ $\sum x_n$ is divergent. #